Fekete-Szegö Functional for Classes $X_q^n(\varphi)$ and $Y_q^n(\varphi)$

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Abstract Two new subclasses of analytic functions are proposed by applying q-differential operator which is denoted as $M_q^n f(z)$. Throughout this study, we acquired the initial coefficients $a_2$ and $a_3$ and the upper bound for the functional $|a_3 - \mu a_2^2|$ of the functions $f$ in the classes $X_q^n(\varphi)$ and $Y_q^n(\varphi)$.

Keywords: Analytic function, Univalent function, q-differential operator, Fekete-Szegö functional, Subordination.

Introduction

The class for all analytic functions $f(z)$ within the open unit disk $\mathbb{U} = \{z: z \in \mathbb{C}, |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$ is represented as $A$. According to Atassi [2], if $f(z)$ has a derivative at each point of $R$ and if $f(z)$ is single valued, then a function $f(z)$ is known to be analytic within region $R$ of the complex plane. Moreover, a function $f(z)$ is known to be analytic at a point $z$ with the condition of $z$ is an interior point of some region where $f(z)$ is analytic. Meanwhile, Kai [9] stated that for each $f \in A$, $f$ has a Taylor series expansion written in the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots = z + \sum_{j=2}^{\infty} a_j z^j, \quad a_j \in \mathbb{C}, z \in \mathbb{U}. \quad (1.1)$$

The definition of subordination according to Jeyaraman & Suresh [6] is as if $f$ and $g$ are in $A$, the function $f$ is said to be subordinate to $g$ or (equivalently) $g$ is said to be superordinate to $f$, $f < g \text{ in } \mathbb{U}$ or $f(z) < g(z) \quad (z \in \mathbb{U})$

if a Schwarz function, $\omega(z)$, analytic in $\mathbb{U}$ with $|\omega(z)| < 1$ and $\omega(0) = 0$ for all $z \in \mathbb{U}$ is exist. For example,

$$f(z) = g(\omega(z)) \quad (z \in \mathbb{U}).$$

In particular, several researchers have done the research about the coefficients, $|a_2|$ and $|a_3|$, and the upper bound for $|a_3 - \mu a_2^2|$ which is known as Fekete-Szegö functional of function $f$. For example, Alsoboh and Darus [1], Aouf and Orhan [3], Janteng et al. [4], Janteng and Halim [5] and Pinhong et al. [11]. Therefore, this study is going to introduce new subclasses of analytic functions and further determine the upper bound for the Fekete-Szegö functional of functions $f$ for particular subclasses of analytic univalent functions which is defined by subordination and $q$-differential operator. Jackson [7] was the earliest researcher developed the $q$-integral and $q$-derivative more systematically.

However, Ramachandran et al. [12] stated that the $q$-derivative operator for function $f$ as
\[ D_qf(z) = \begin{cases} \frac{f(qz) - f(z)}{(q - 1)z}, & z \neq 0, 0 < q < 1 \\ f'(0), & z = 0 \end{cases} \]

for functions \( f \) which are differentiable at \( z = 0 \).

Then, Koekoek and Koekoek [8] further defined \( D_q^nf \) as

\[ D_q^nf = D_q(D_q^{n-1}f) \]

for \( n = 1, 2, 3, \ldots \) where \( D_q^0 \) denotes the identity operator.

For the used of \( D_qf(z) \), Seoudy and Aouf [13] introduced the subclasses \( S_q^\alpha(A) \) and \( C_q^\alpha(A) \) of the class \( A \) for \( 0 \leq \alpha < 1 \) which are defined by

\[ S_q^\alpha(A) = \left\{ f \in A : \frac{zqD_qf(z)}{f(z)} > \alpha, z \in \mathbb{U} \right\}, \]

\[ C_q^\alpha(A) = \left\{ f \in A : \frac{D_q\left(zD_qf(z)\right)}{D_qf(z)} > \alpha, z \in \mathbb{U} \right\}. \]

Selvaraj et al. [14] noted that

\[ f \in C_q^\alpha(A) \iff zD_qf \in S_q^\alpha(A), \]

Alsoboh and Darus [1] proposed \( q \)-differential operator of a function \( f \) in the form of (1.1) and denoted by \( M_q^\alpha(f) \) as

\[ M_q^\alpha(f)(z) = f(z), \quad M_q^\alpha(zf)'(z) = zD_qf(z) = z + \sum_{j=2}^\infty j \eta [j]_q z^j, \]

\[ M_q^\alpha(zD_qf(z)) = zD_q\left(M_q^{\alpha-1}f(z)\right) = z + \sum_{j=2}^\infty \eta [j]_q z^j \tag{1.2} \]

where \([j]_q \equiv \frac{1-q^j}{1-q}\) which was defined by Jackson [7].

By using the \( q \)-differential operator in (1.2) and the principle of subordination, we propose two new subclasses, \( X_q^n(\varphi) \) and \( Y_q^n(\varphi) \), of \( A \).

Let \( P \) to be denoted as class of all functions \( \varphi \) that is analytic and univalent in \( \mathbb{U} \).

The definitions of classes \( X_q^n(\varphi) \) and \( Y_q^n(\varphi) \) where \( \varphi \in P \) are given respectively.

**Definition 1.1** A function \( f \in A \) is categorized in the class \( X_q^n(\varphi) \) if the following subordination condition hold

\[ D_q\left(M_q^n f(z)\right) < \varphi(z), \quad \varphi \in P, n \in N, 0 < q < 1, z \in \mathbb{U}. \]

**Definition 1.2** A function \( f \in A \) is categorized in the class \( Y_q^n(\varphi) \) if the following subordination condition hold

\[ (1 - \delta) zD_q(M_q^n f(z)) + \delta \left( 1 + \frac{qzD_q\left(D_q M_q^n f(z)\right)}{D_q(M_q^n f(z))} \right) < \varphi(z), \quad \varphi \in P, n \in N, 0 < q < 1, 0 \leq \delta \leq 1 \text{ and } z \in \mathbb{U}. \]

Next, the lemma that is used to validate the main results in order to get the upper bound for the Fekete-Szegő functional for \( f \in X_q^n(\varphi) \) and \( f \in Y_q^n(\varphi) \) is as below.

**Lemma 1.1** ([10]) If \( p(z) = 1 + c_1z + c_2z^2 + \cdots \) is a function with positive real part in \( \mathbb{U} \) and \( \gamma \) is a complex number, then

\[ |c_2 - \gamma c_1^2| \leq 2 \max\{1, |2\gamma - 1|\}. \]

The result is sharp for the functions given by
\[ p(z) = \frac{1 + z^2}{1 - z^2} \quad \text{and} \quad p(z) = \frac{1 + z}{1 - z}. \]

**Main Results**

**Theorem 2.1**  
Let \( \varphi(z) = 1 + B_1z + B_2z^2 + B_3z^3 \ldots \) with \( B_1 \neq 0 \), and \( f \) is given by (1.1) be in the class \( X^n_q(\varphi) \) and \( \mu \) is a complex number, then

\[ |a_3 - \mu a_2^2| \leq \frac{B_1}{[3]_q^{n+1}} \max \left\{ 1, \left| \frac{B_2}{B_1} - \frac{[3]_q^{n+1} B_1}{[2]_q^{2n+2}} \right| \right\} \]

**Proof.**  
If \( f \in X^n_q(\varphi) \), then Schwarz function \( \omega(z) \) is exist with \( \omega(0) = 0 \) and \( |\omega(z)| < 1 \) in \( \mathbb{U} \) such that

\[ D_q \left( M^n_q f(z) \right) = \varphi(\omega(z)). \]  

(2.1)

The function \( p(z) \) is defined as

\[ p(z) = \frac{1 + \omega(z)}{1 - \omega(z)} = 1 + p_1z + p_2z^2 + \ldots \]  

(2.2)

We see that \( \text{Re}(p(z)) > 0 \) and \( p(0) = 1 \) with \( \omega(z) \) as Schwarz function. Let

\[ g(z) = D_q \left( M^n_q f(z) \right) = 1 + d_1z + d_2z^2 + \ldots \]  

(2.3)

From equations (2.1), (2.2) and (2.3), we get that

\[ g(z) = \varphi \left( \frac{p(z) - 1}{p(z) + 1} \right) \]

By equation (2.2), we solve \( \omega(z) \) in terms of \( p(z) \), we get that

\[ \omega(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{p_1z + p_2z^2 + \ldots}{2 + p_1z + p_2z^2 + \ldots}. \]

where

\[ \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2} \left( p_1z + \left( p_2 - \frac{p_1^2}{2} \right) z^2 + \left( p_3 + \frac{p_1 p_2}{4} - p_1 p_2 \right) z^3 + \ldots \right). \]  

(2.4)

From equations \( \varphi(z) \) and (2.4), we get that

\[ g(z) = \varphi \left( \frac{1}{2} \left( p_1z + \left( p_2 - \frac{p_1^2}{2} \right) z^2 + \left( p_3 + \frac{p_1 p_2}{4} - p_1 p_2 \right) z^3 + \ldots \right) \right) \]

\[ = 1 + \frac{1}{2} B_1 \left( \frac{1}{2} \left( p_1z + \left( p_2 - \frac{p_1^2}{2} \right) z^2 + \ldots \right) \right)^2 + \ldots \]

\[ = 1 + \frac{1}{2} B_1 p_1z + \left( \frac{1}{2} B_1 \left( p_2 - \frac{p_1^2}{2} \right) + \frac{1}{4} B_2 p_1^2 \right) z^2 + \ldots. \]  

(2.5)

From (2.3) and (2.5), we obtain

\[ d_1 = \frac{1}{2} B_1 p_1, \quad \text{and} \quad d_2 = \frac{1}{2} B_1 \left( p_2 - \frac{p_1^2}{2} \right) + \frac{1}{4} B_2 p_1^2. \]

From (1.2), a computation shows that

\[ M^n_q f(z) = z + [2]_q a_2 z^2 + [3]_q a_3 z^3 + \ldots. \]  

(2.6)

According to the definition of \( D_q f \) stated by Ramachandran et al. [12], we obtain
\[ D_q \left( M_q^n f(z) \right) = 1 + (q + 1)[2]_q^n a_2 z + (q^2 + q + 1)[3]_q^n a_3 z^2 + \ldots \]  \hspace{1cm} (2.7)

According to the definition of \([j]_q\) by Jackson [7], let \(j = 0, 1, 2\) and 3, we obtain that

when \(j = 0\),
\[ [0]_q = \frac{1 - q^0}{1 - q} = 0 \]

when \(j = 1\),
\[ [1]_q = \frac{1 - q^1}{1 - q} = 1 \]

when \(j = 2\),
\[ [2]_q = \frac{1 - q^2}{1 - q} = 1 + q \]  \hspace{1cm} (2.8)

when \(j = 3,\)
\[ [3]_q = \frac{1 - q^3}{1 - q} = q^2 + q + 1 \]  \hspace{1cm} (2.9)

Substitute (2.8) and (2.9) into (2.7), we obtain

\[ D_q \left( M_q^n f(z) \right) = 1 + [2]_q^{n+1} a_2 z + [3]_q^{n+1} a_3 z^2 + \ldots \]  \hspace{1cm} (2.10)

Then, compared (2.3) to (2.10), we obtain
\[ d_1 = [2]_q^{n+1} a_2 \]
and
\[ d_2 = [3]_q^{n+1} a_3 \]
or equivalently we have
\[ d_1 = \frac{1}{2} B_1 p_1 = [2]_q^{n+1} a_2, \]
\[ a_2 = \frac{B_1 p_1}{2[2]_q^{n+1}} \]
and
\[ d_2 = \frac{1}{2} B_1 \left( p_2 - \frac{p_1^2}{2} \right) + \frac{1}{4} B_2 p_1^2 = [3]_q^{n+1} a_3, \]
\[ a_3 = \frac{B_1}{2[3]_q^{n+1}} \left( p_2 - \frac{p_1^2}{2} \right) + \frac{B_2 p_1^2}{4[3]_q^{n+1}}. \]

Now,
\[ a_3 - \mu a_2^2 = \frac{B_1}{2[3]_q^{n+1}} \left( p_2 - \frac{p_1^2}{2} \right) + \frac{B_2 p_1^2}{4[3]_q^{n+1}} - \mu \left( \frac{B_1 p_1}{2[2]_q^{n+1}} \right)^2, \]
\[ a_3 - \mu a_2^2 = \frac{B_1 p_2}{2[3]_q^{n+1}} - \frac{B_2 p_1^2}{4[3]_q^{n+1}} + \frac{B_2 p_1^2}{4[3]_q^{n+1}} - \frac{B_2 p_1^2}{4[3]_q^{n+1}}, \]
\[ a_3 - \mu a_2^2 = \frac{B_1}{2[3]_q^{n+1}} \left( p_2 - \frac{p_1^2}{2} + \frac{B_2 p_1^2}{2B_1} \right) - \frac{B_2 p_1^2}{2B_1} - \frac{[3]_q^{n+1} \mu B_1}{2[2]_q^{n+1}}, \]
\[ a_3 - \mu a_2^2 = -\frac{B_1}{2[3]_q^{n+1}} \left( p_2 - \frac{p_1^2}{2} - \frac{B_2 p_1^2}{2B_1} - \frac{[3]_q^{n+1} \mu B_1}{2[2]_q^{n+1}} \right). \]

consider
\[ \gamma = \frac{1}{2} \left( 1 - \frac{B_2 p_1^2}{B_1} + \frac{[3]_q^{n+1} \mu B_1}{[2]_q^{n+1}} \right). \]

Therefore,
\[ a_3 - \mu a_2^2 = -\frac{B_1}{2[3]_q^{n+1}} \left( p_2 - \gamma p_1^2 \right). \]

By applying Lemma 1.1, it shows that
\[ |a_3 - \mu a_2^2| \leq \frac{B_1}{|3_q|^{n+1}} \max \left\{ 1; \frac{B_2}{B_1} - \frac{|3|^{q+1} \mu B_1}{|2|^{2n+2}} \right\} \]

The proof of Theorem 2.1 is done.

Taking \( n = 0 \) into Theorem 2.1, we acquire the corollary below.

**Corollary 2.1**  
Let \( \varphi(z) = 1 + B_1z + B_2z^2 + B_3z^3 \ldots \) with \( B_1 \neq 0 \), and \( f \) is given by (1.1) be in the class \( X_q(\varphi) \) and \( \mu \) is a complex number, then

\[ |a_3 - \mu a_2^2| \leq \frac{B_1}{|3_q|} \max \left\{ 1; \frac{B_2}{B_1} - \frac{|3_q| \mu B_1}{|2_q|} \right\}. \]

Now, we show the results for class \( Y_q^n(\varphi) \).

**Theorem 2.2**  
Let \( \varphi(z) = 1 + B_1z + B_3z^2 + B_3z^3 \ldots \) with \( B_1 \neq 0 \), and \( f \) is given by (1.1) be in the class \( Y_q^n(\varphi) \) and \( \mu \) is a complex number, then

\[ |a_3 - \mu a_2^2| \leq \frac{B_1}{2 \left( 1 - \delta \right)(|3_q^n|(|3_q| - 1) + \delta q|3|^{n+2})} \max \left\{ 1; \frac{B_2}{B_1} - \frac{\delta |3_q^n|(|3_q| - 1) + \delta q|3|^{n+3} - \mu \left( (1 - \delta)(|3_q^n|(|3_q| - 1) + \delta q|3|^{n+2}) \right)}{\delta |3_q^n|(|3_q| - 1) + \delta q|3|^{n+3} - \mu \left( (1 - \delta)(|3_q^n|(|3_q| - 1) + \delta q|3|^{n+2}) \right)} \right\}. \]

**Proof.**  
If \( f \in Y_q^n(\varphi) \), then Schwarz function \( \omega(z) \) is exist with \( \omega(0) = 0 \) and \( |\omega(z)| < 1 \) in \( \mathbb{U} \) such that

\[ (1 - \delta) \frac{z d_q^\omega f(z)}{M_q^\omega f(z)} + \delta \left( 1 + \frac{z d_q^\omega f(z)}{M_q^\omega f(z)} \right) = \varphi(\omega(z)). \]  \hspace{1cm} (2.11)

We see that \( Re(p(z)) > 0 \) and \( p(0) = 1 \) with \( \omega(z) \) as Schwarz function. Let

\[ g(z) = (1 - \delta) \frac{z d_q^\omega f(z)}{M_q^\omega f(z)} + \delta \left( 1 + \frac{z d_q^\omega f(z)}{M_q^\omega f(z)} \right) = 1 + d_1z + d_2z^2 + \ldots \]  \hspace{1cm} (2.12)

From equations (2.2), (2.11) and (2.12), we get

\[ g(z) = \varphi \left( \frac{p(z) - 1}{p(z) + 1} \right). \]

By equation (2.2), we solve \( \omega(z) \) in terms of \( p(z) \), we get

\[ \omega(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{p_1z + p_2z^2 + \ldots}{1 + 2p_1z + p_2z^2 + \ldots}. \]

Where

\[ \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2} \left( p_1z + \left( p_2 - \frac{p_1^2}{2} \right) z^2 + \left( p_3 + \frac{p_1^3}{4} - p_1p_2 \right) z^3 + \ldots \right). \]  \hspace{1cm} (2.13)

From equations \( \varphi(z) \) and (2.13), we get

\[ g(z) = \varphi \left( \frac{p(z) - 1}{p(z) + 1} \right) \]

\[ = \varphi \left( \frac{1}{2} \left( p_1z + \left( p_2 - \frac{p_1^2}{2} \right) z^2 + \left( p_3 + \frac{p_1^3}{4} - p_1p_2 \right) z^3 + \ldots \right) \right) \]

\[ = 1 + B_1 \left( \frac{1}{2} \left( p_1z + \left( p_2 - \frac{p_1^2}{2} \right) z^2 + \ldots \right) \right)^2 + \ldots \]
\[ = 1 + \frac{1}{2} B_1 p_1 z + \left(\frac{1}{2} B_1 \left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4} B_2 p_1^2\right) z^2 + \ldots. \quad (2.14) \]

From (2.2) and (2.14), we obtain
\[ d_1 = \frac{1}{2} B_1 p_1, \]
and
\[ d_2 = \frac{1}{2} B_1 \left(p_2 - \frac{p_1^2}{2}\right) + \frac{1}{4} B_2 p_1^2. \]

Case 1 for the equation \((1 - \delta) \frac{zD_q(M_q^n f(z))}{M_q^n f(z)}\), we substitute (2.6) and (2.10) into the equation and obtain
\[
(1 - \delta) \frac{zD_q(M_q^n f(z))}{M_q^n f(z)} = (1 - \delta) \left(\frac{z(1 + [2]_q^n a_2 z + [3]_q^n a_3 z^2 + \ldots)}{z + [2]_q^n a_2 z^2 + [3]_q^n a_3 z^3 + \ldots}\right)
\]
\[= (1 - \delta)(1 + [2]_q^n([2]_q - 1)a_2 z + ([3]_q^n([3]_q - 1)a_3 - [2]_q^n([2]_q - 1) a_2^2) z^2 + \ldots) \quad (2.15)\]

Case 2 for the equation \(\delta \left(1 + \frac{qzD_q(D_qM_q^n f(z))}{D_q(M_q^n f(z))}\right)\), by Alsoboh and Darus [1],
\[
\delta \left(1 + \frac{qzD_q(D_qM_q^n f(z))}{D_q(M_q^n f(z))}\right) = \delta(1 + q a_2 [2]_q^{3+2}z + q(a_3[3]^{n+2} - a_2^2[2]_q^{2n+3}) z^2 + \ldots) \quad (2.16)\]

Therefore, a computation of (2.15) and (2.16) shows that
\[
(1 - \delta) \frac{zD_q(M_q^n f(z))}{M_q^n f(z)} + \delta \left(1 + \frac{qzD_q(D_qM_q^n f(z))}{D_q(M_q^n f(z))}\right)
\]
\[= (1 - \delta)(1 + [2]_q^n([2]_q - 1)a_2 z + ([3]_q^n([3]_q - 1)a_3 - [2]_q^n([2]_q - 1) a_2^2) z^2 + \ldots)
\]
\[+ \delta(1 + q a_2 [2]_q^{3+2}z + q(a_3[3]^{n+2} - a_2^2[2]_q^{2n+3}) z^2 + \ldots)
\]
\[= 1 + \left((1 - \delta)[2]_q^n([2]_q - 1)a_2 + \delta q a_2 [2]_q^{3+2} z + (1 - \delta)([3]_q^n([3]_q - 1)a_3 - (1 - \delta)[2]_q^n([2]_q - 1)a_2^2 + \delta q(a_3[3]^{n+2} - a_2^2[2]_q^{2n+3})\right) z^2 + \ldots \quad (2.17)\]

Then, compared (2.14) to (2.17), we get
\[d_1 = (1 - \delta)[2]_q^n([2]_q - 1)a_2 + \delta q a_2 [2]_q^{3+2},\]
\[d_1 = a_2 \left((1 - \delta)[2]_q^n([2]_q - 1) + \delta q [2]_q^{3+2}\right) \]

and
\[d_2 = (1 - \delta)([3]_q^n([3]_q - 1)a_3 - (1 - \delta)[2]_q^n([2]_q - 1)a_2^2 + \delta q(a_3[3]^{n+2} - a_2^2[2]_q^{2n+3}),\]
\[d_2 = a_3 \left((1 - \delta)([3]_q^n([3]_q - 1) + \delta q[3]^{n+2}) - a_2^2((1 - \delta)[2]_q^n([2]_q - 1) + \delta q [2]_q^{3+2}\right) \]

or equivalently we have
\[d_1 = \frac{1}{2} B_1 p_1 = a_2 \left((1 - \delta)[2]_q^n([2]_q - 1) + \delta q [2]_q^{3+2}\right),\]
\[B_1 p_1 = \frac{a_2}{2 \left((1 - \delta)[2]_q^n([2]_q - 1) + \delta q [2]_q^{3+2}\right)}, \]

and

\[10.11113/mjfas.v20n2.3228] 440
\[ d_2 = \frac{1}{2} B_1 \left( p_2 - \frac{p_1^2}{2} \right) + \frac{1}{4} B_2 p_1^2 \]
\[ = a_3 ((1 - \delta)([3]_{\bar{m}})^n([3]_q - 1) + \delta q [3]^{n+2}) - a_3^2 ((1 - \delta)[2]_{\bar{m}}^n([2]_q - 1) + \delta q [2]_{\bar{m}}^{n+3}). \]

\[ a_3 (1 - \delta)([3]_{\bar{m}})^n([3]_q - 1) + \delta q [3]^{n+2} \]
\[ = \left( \frac{B_1 p_1}{2 \left( (1 - \delta)[2]_{\bar{m}}^n([2]_q - 1) + \delta q [2]_{\bar{m}}^{n+2} \right)} \right)^2 \left( (1 - \delta)[2]_{\bar{m}}^n([2]_q - 1) + \delta q [2]_{\bar{m}}^{n+3} \right) \]
\[ + \frac{1}{2} B_1 \left( p_2 - \frac{p_1^2}{2} \right) + \frac{1}{4} B_2 p_1^2, \]

\[ a_3 = \frac{1}{(1 - \delta)([3]_{\bar{m}})^n([3]_q - 1) + \delta q [3]^{n+2}} \left( \frac{B_1^2 p_1^2 ((1 - \delta)[2]_{\bar{m}}^n([2]_q - 1) + \delta q [2]_{\bar{m}}^{n+3})}{4 \left( (1 - \delta)[2]_{\bar{m}}^n([2]_q - 1) + \delta q [2]_{\bar{m}}^{n+2} \right)^2} \right) \]
\[ + \frac{1}{2} B_1 \left( p_2 - \frac{p_1^2}{2} \right) + \frac{1}{4} B_2 p_1^2. \]

Now,

\[ a_3 - \mu a_3^2 = \frac{1}{(1 - \delta)([3]_{\bar{m}})^n([3]_q - 1) + \delta q [3]^{n+2}} \left( \frac{B_1^2 p_1^2 ((1 - \delta)[2]_{\bar{m}}^n([2]_q - 1) + \delta q [2]_{\bar{m}}^{n+3})}{4 \left( (1 - \delta)[2]_{\bar{m}}^n([2]_q - 1) + \delta q [2]_{\bar{m}}^{n+2} \right)^2} \right) \]
\[ + \frac{1}{2} B_1 \left( p_2 - \frac{p_1^2}{2} \right) + \frac{1}{4} B_2 p_1^2 - \mu \left( \frac{B_1 p_1}{2 \left( (1 - \delta)[2]_{\bar{m}}^n([2]_q - 1) + \delta q [2]_{\bar{m}}^{n+2} \right)} \right)^2 \]
\[ - \delta [2]_{\bar{m}}^n([2]_q - 1) + \delta q [2]_{\bar{m}}^{n+3} - \mu \left( (1 - \delta)([3]_{\bar{m}})^n([3]_q - 1) + \delta q [3]^{n+2} \right) \]
\[ + \left( \frac{B_1 p_2 - B_1 p_1^2}{2} + \frac{1}{4} B_2 p_1^2 \right), \]

\[ a_3 - \mu a_3^2 = \frac{1}{(1 - \delta)([3]_{\bar{m}})^n([3]_q - 1) + \delta q [3]^{n+2}} \left( p_2 \left( \frac{B_1}{2} \right) \right) \]
\[ + p_1^2 \left( \frac{B_1}{2 \left( (1 - \delta)[2]_{\bar{m}}^n([2]_q - 1) + \delta q [2]_{\bar{m}}^{n+2} \right)} \right)^2 \left( (1 - \delta)[2]_{\bar{m}}^n([2]_q - 1) + \delta q [2]_{\bar{m}}^{n+3} \right) \]
\[ - \mu \left( (1 - \delta)([3]_{\bar{m}})^n([3]_q - 1) + \delta q [3]^{n+2} \right) \right) \left( \frac{B_2 - B_1}{4} \right). \]
\[
\alpha_3 - \mu \alpha_2^2 = \frac{B_1}{2 (1 - \delta)([3]_q^n)([3]_q - 1) + \delta q[3]^{n+2}} \left( p_2 - p_1^2 \right) - \frac{B_1}{2 \left( 1 - \delta \right) [2]^n ([2]_q - 1) + \delta q[2]^{n+2}} \left( 1 - \frac{B_2}{2B_1} - \frac{1}{2} \right).
\]

Consider
\[
\gamma = - \frac{B_1}{2 \left( 1 - \delta \right) [2]^n ([2]_q - 1) + \delta q[2]^{n+2}} \left( 1 - \frac{B_2}{2B_1} - \frac{1}{2} \right).
\]

Therefore,
\[
\alpha_3 - \mu \alpha_2^2 = \frac{B_1}{2 \left( 1 - \delta \right) [3]_q^n ([3]_q - 1) + \delta q[3]^{n+2}} \left( p_2 - \gamma p_1^2 \right).
\]

By applying Lemma 1.1, it shows that
\[
\left| \alpha_3 - \mu \alpha_2^2 \right| \leq \frac{B_1}{2 \left( 1 - \delta \right) [3]_q^n ([3]_q - 1) + \delta q[3]^{n+2}} \left( \max \left\{ 1; \frac{B_1}{(1 - \delta) [2]^n ([2]_q - 1) + \delta q[2]^{n+2}} \left( 1 - \delta \right) [2]_q^n ([2]_q - 1) + \delta q[2]^{n+3} - \mu \left( 1 - \delta \right) [3]_q^n ([3]_q - 1) + \delta q[3]^{n+2} \right) - \frac{B_2}{B_1} \right) \right).
\]

The proof of Theorem 2.2 is done.

Taking \( \delta = 1 \) into Theorem 2.2, we acquire the corollary below.

**Corollary 2.2** \(([1])\) Let \( \varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 \ldots \) with \( B_1 \neq 0 \), and \( f \) is given by (1.1) be in the class \( Y_q^n(\varphi) \) and \( \mu \) is a complex number, then
\[
\left| \alpha_3 - \mu \alpha_2^2 \right| \leq \frac{B_1}{2q[3]^{n+2}} \max \left\{ 1; \frac{B_2}{B_1} + \frac{1}{[2]_q} - \mu \left( \frac{[3]_q^{n+2}}{[2]_q^{n+4} q} \right) \right\}.
\]

**Conclusions**

In conclusion, we acquired the initial coefficients \( \alpha_2 \) and \( \alpha_3 \) and the upper bound for the functional \( \left| \alpha_3 - \mu \alpha_2^2 \right| \) of the functions \( f \) in the class \( X_q^n(\varphi) \) and class \( Y_q^n(\varphi) \).
Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

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