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Ideal convergence in n-normal spaces and some new sequence spaces via n-norm

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ABSTRACT

In this paper we introduced some new sequence spaces using n-normed spaces and gave some preliminary result for matrix transformations between some sequence spaces.

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| Natural density | Statistical convergence | Statistical cauchy sequence | $\mathcal{I}$-convergence | 2-normed spaces | n-normed spaces |

1. Introduction

P. Kostyrko et al [**11**] introduced the concept of $\mathcal{I}$-convergence of sequences in a metric space and studied some properties of such convergence. Note that $\mathcal{I}$-convergence is an interesting generalization of statistical convergence.

The concept of statistical convergence was introduced by Steinhaus [**19**] in 1951 (see also Fast [**1**]) and had been discussed and developed by many authors including [**2**], [**3**], [**15**].

Let $\mathbb{N}$ denotes the set of positive integers and $(X, \|.\|)$ be a normed space. Recall that a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X is called to be statistically convergent to $x \in X$ if the set $A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - x\| \geq \varepsilon\}$ has natural density zero for each $\varepsilon > 0$.

Let us now give some definitions and notations.

A family $\mathcal{I} \subset 2^Y$ of subsets a nonempty set Y is said to be an ideal in Y if (**i**) $\emptyset \in \mathcal{I}$; (**ii**) $A, B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$; (**iii**) $A \in \mathcal{I}$, $B \subset A$ imply $B \in \mathcal{I}$, while an admissible ideal $\mathcal{I}$ of Y further satisfies $\{x\} \in \mathcal{I}$ for each $x \in Y$ [**10**], [**13**].

Let $Y \neq \emptyset$. A non-empty family $\mathcal{F} \subset 2^Y$ is said to be a filter on Y if (i) $\emptyset \notin \mathcal{F}$;

(ii) $A, B \in \mathcal{F}$ imply $A \cap B \in \mathcal{F}$; (iii) $A \in \mathcal{F}$, $A \subset B \subset Y$ imply $B \in \mathcal{F}$. Let $\mathcal{I}$ be a proper ideal in Y (i.e. $Y \notin \mathcal{I}$), $Y \neq \emptyset$. Then the family of sets $\mathcal{F}(\mathcal{I}) = \{M \subset Y : \exists A \in \mathcal{I} : M = Y \backslash A\}$ is a filter in Y. It is called the filter associated with the ideal $\mathcal{I}$ [**12**].

Given $\mathcal{I} \subset 2^{\mathbb{N}}$ be a nontrivial ideal in $\mathbb{N}$. The sequence $(x_n)_{n \in \mathbb{N}}$ in X is said to be $\mathcal{I}$-convergent to $x \in X$, if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - x\| \geq \varepsilon\}$ belongs to $\mathcal{I}$ [**11, 12**]. There are many examples to ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ in [**11, 12**] and basic properties of $\mathcal{I}$-convergence have been studied in these works.

Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a nontrivial ideal in $\mathbb{N}$ and $(X, \|.\|)$ be a normed space. The sequence $x = (x_n)$ of elements of X is said to be $\mathcal{I}$-convergence to $x \in X$ if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in \mathbb{N} : \|x_n - x\| \geq \varepsilon\}$ belongs to $\mathcal{I}$. If $x = (x_n)$ is $\mathcal{I}$-convergent to x then we write $\mathcal{I}\text{-}\lim_{n \to \infty} x_n = x$. In this case the element $x \in X$ is called $\mathcal{I}$-limit of the sequence $x = (x_n) \in X$ [**11, 12**].

There are many examples to ideals $\mathcal{I} \subset 2^{\mathbb{N}}$ in [**11, 12**] and basic properties of $\mathcal{I}$-convergence have been studied in these works.

The concept of 2-normed spaces was initially introduced by Gähler [**4**] in the 1960's. Since then, many researchers have studied this concept and obtained various results, see for instance [**5, 7, 17, 18**].

Let $n \in \mathbb{N}$ and X be a real vector space of dimension d, where $n \leq d$. An n-norm on X is a function $\|., ..., .\| : \underbrace{X \times X \times ... \times X}_{n-times} \to \mathbb{R}$ which satisfies the following four conditions:

(**i**) $\|x_1, x_2, ..., x_n\| = 0$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent;

(**ii**) $\|x_1, x_2, ..., x_n\|$ are invariant under permutation;

(**iii**) $\|\alpha x_1, x_2, ..., x_n\| = |\alpha| \, \|x_1, x_2, ..., x_n\|$, $\alpha \in \mathbb{R}$;

(**iv**) $\|x + x', x_2, ..., x_n\| \leq \|x, x_2, ..., x_n\| + \|x', x_2, ..., x_n\|$.

The pair $(X, \|., ..., .\|)$ is then called a n-normed space [**6**].

Definition 1. [**8**] *A sequence (x_k) in n-normed space $(X, \|., ..., .\|)$ is said to be convergent to an x in X (in the n-norm) if*

$$\lim_{k \to \infty} \|x_1, x_2, ..., x_{n-1}, x_k - x\| = 0$$

for every $x_1, x_2, ..., x_{n-1} \in X$.

Definition 2. [**5**] *A sequence (x_k) in n-normed space $(X, \|., ..., .\|)$ is said to be Cauchy in X (with respect to the n-norm)if*

$$\lim_{k,l \to \infty} \|x_1, x_2, ..., x_{n-1}, x_k - x_l\| = 0$$

for every $x_1, x_2, ..., x_{n-1} \in X$.

If every Cauchy sequence converges to an x in X then X is said to be complete (with respect to $n-$ norm). Any complete n-normed space is said to be n-Banach space.

Let $(X, \|., ..., .\|)$ is an n-normed space of dimension $d \geq n$ and $\{a_1, a_2, ..., a_n\}$ is a linearly independent set in X. Then the function $\|., ..., .\|_\infty$ on X^{n-1} defined by

$$\|x_1, x_2, ..., x_{n-1}\|_\infty := \max\{\|x_1, x_2, ..., x_{n-1}, a_i\| : i = 1, ..., n\}$$

defines an $(n-1)$ norm on X with respect to $\{a_1, a_2, ..., a_n\}$ [**5**].

Definition 3. [**16**] *Let X be a linear space. Then a map $g : X \to \mathbb{R}$ is called a paranorm (on X) if is satisfies the following conditions for all $x, y \in X$:*

(i) $g(\theta) = 0$ (Here $\theta = (0, 0, ..., 0, ...)$ is zero of the space)

(ii) $g(x) = g(-x)$

(iii) $g(x + y) \leq g(x) + g(y)$

(iv) $g(\lambda^n - \lambda) \to 0$ $(n \to \infty)$ and $g(x^n - x) \to 0$ $(n \to \infty)$ imply $g(\lambda^n x^n - \lambda x) \to 0$ $(n \to \infty)$.

Recall that $(X, \|., ..., .\|)$ is a n-Banach space if every Cauchy sequence in X is convergent to some x in X in the n-norm.

The following lemma will help us throughout our study.

Lemma 1. [**8**] *$(X, \|., ..., .\|)$ is a n-Banach space if and only if $(X, \|., ..., .\|)$ is a Banach space.*

Now, we introduce the notion of $\mathcal{I}$-convergence in n-normed spaces and give the main results of the paper.

2. Ideal Convergence of n-Normed Spaces

Suppose hereafter that $(X, \|., ..., .\|)$ is n-normed space. Recall that we assume X to have dimension d, where $2 \leq n \leq d < \infty$, unless otherwise stated.

Definition 4. *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a nontrivial ideal in $\mathbb{N}$. The sequence (x_k) of X is said to be $\mathcal{I}$-convergent to x, if for each $\varepsilon > 0$ and $x_1, x_2, ..., x_{n-1}$ in X the set $A(\varepsilon) = \{k \in \mathbb{N} : \|x_k - x, x_1, x_2, ..., x_{n-1}\| \geq \varepsilon\}$ belongs to $\mathcal{I}$.*

If (x_n) is $\mathcal{I}$-convergent to x then we write $\mathcal{I}\text{-}\lim_{n\to\infty} \|x_k - x, x_1, x_2, ..., x_{n-1}\| = 0$ or $\mathcal{I}\text{-}\lim_{k\to\infty} \|x_k, x_1, x_2, ..., x_{n-1}\| = \|x, x_1, x_2, ..., x_{n-1}\|$. The number x is $\mathcal{I}$-limit of the sequence (x_k).

Further we will give some examples of ideals and corresponding $\mathcal{I}$-convergences.

(I) Let $\mathcal{I}_f$ be the family of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}_f$ is an admissible ideal in $\mathbb{N}$ and $\mathcal{I}_f$ convergence coincides with usual convergence in [**4**] .

(II) Put $\mathcal{I}_\delta = \{A \subset \mathbb{N} : \delta(A) = 0\}$. Then $\mathcal{I}_\delta$ is an admissible ideal in $\mathbb{N}$ and $\mathcal{I}_\delta$ convergence coincides with the statistical convergence in [**9**].

Remark 1. *Note that if $\mathcal{I}$ is admissible ideal, then the convergence in n-normed space $(X, \|.,...,.\|)$ implies $\mathcal{I}$-convergence in n-normed space $(X, \|.,...,.\|)$.*

We shall now investigate which axioms of convergence in X are satisfied by $\mathcal{I}$-convergence in X.

(S) Every constant sequence $(x, x, ..., x, ...)$ converges to x in n-normed space X.

(H) The limit of any convergent sequence in n-normed space X is uniquely determined.

(F) If a sequence $(x_k)_{k\in\mathbb{N}}$ in X has the limit x in X, then each of its subsequences has the same limit.

(U) If each subsequence of the sequence $(x_k)_{k\in\mathbb{N}}$ in X has a subsequence which converges to x in X, then (x_k) converges to x in X.

Proposition 1. *Suppose that n-normed space X has at least two points. Let $\mathcal{I} \subset 2^X$ be an admissible ideal.*

(i) The $\mathcal{I}$-convergence in X satisfies (S),(H) and (U).

(ii) If $\mathcal{I}$ contains an infinite set, then $\mathcal{I}$-convergence in X does not satisfy (F).

Proof. (i) (S) is obviously fulfilled. To prove (H) it is sufficient to observe that for any $A_1, A_2 \in \mathcal{I}$ we have $(\mathbb{N}\backslash A_1) \cap (\mathbb{N}\backslash A_2) \neq \emptyset$ since the last two sets belong to the filter associated with $\mathcal{I}$. If there are two limits $x_1, x_2 \in X$, $x_1 \neq x_2$, choose ε such that

$$0 < 2\varepsilon < \|x_1 - x_2, x_1, x_2, ..., x_{n-1}\|$$

where $x_1 - x_2$ and $x_1, x_2, ..., x_{n-1}$ are linearly independent. And put

$$\begin{aligned} A_1 &= \{k \in \mathbb{N} : \|x_k - x_1, x_1, x_2, ..., x_{n-1}\| \geq \varepsilon\}, \\ A_2 &= \{k \in \mathbb{N} : \|x_k - x_2, x_1, x_2, ..., x_{n-1}\| \geq \varepsilon\}. \end{aligned}$$

Suppose now that (U) does not hold. Then there exists $\varepsilon_0 > 0$ such that

$$A(\varepsilon_0) = \{k \in \mathbb{N} : \|x_k - x, x_1, x_2, ..., x_{n-1}\| \geq \varepsilon_0\} \notin \mathcal{I}$$

for each $x_1, x_2, ..., x_{n-1}$ in X. But then $A(\varepsilon_0)$ is an infinite set since $\mathcal{I}$ is admissible. Let $A(\varepsilon_0) = \{n_1 < n_2 < ... < n_i < ...\}$. Put $y_i = x_{n_i}$ for $i \in \mathbb{N}$. Then $\{y_i\}_{i\in\mathbb{N}}$ is a subsequence of (x_k) without a subsequence $\mathcal{I}$-convergence to x in $n-$normed space X.

(ii) Suppose that $A \in \mathcal{I}$ is an infinite set, $A = \{k_1 < k_2 < ... < k_i < ...\}$. $B = \mathbb{N} \backslash A = \{m_1 < m_2 < ... < m_i < ...\}$. The set B is also infinite since $\mathcal{I}$ is non-trivial ideal. Define (x_k) by choosing $x_1, x_2 \in X$, $x_1 \neq x_2$ and put $x_{k_i} = x_1$, $x_{m_i} = x_2$ for $i \in \mathbb{N}$. Obviously $\mathcal{I}\text{-}\lim_{i\to\infty} \|x_{k_i}, x_1, x_2, ..., x_{n-1}\| = \|z_1, x_1, x_2, ..., x_{n-1}\|$ and $\mathcal{I}\text{-}\lim_{i\to\infty} \|x_{m_i}, x_1, x_2, ..., x_{n-1}\| = \|z_2, x_1, x_2, ..., x_{n-1}\|$. □

Now Let $X = \mathbb{R}^d$ $(d \geq n)$ be equipped with the n-norm then

$\|x_1, x_2, ..., x_{n-1}, x_n\|_S :=$ the volume of the n-dimensional parallelepiped spanned

by the vectors , $x_1, x_2, ..., x_{n-1}, x_n$ which may be given explicitly by the formula

$$\|x_1, x_2, ..., x_{n-1}, x_n\|_S = \begin{vmatrix} < x_1, x_2 > & \cdots & < x_1, x_n > \\ \vdots & \ddots & \vdots \\ < x_n, x_1 > & \cdots & < x_n, x_n > \end{vmatrix}^{\frac{1}{2}}$$

Example 1. *Let $\mathcal{I} = \mathcal{I}_\delta$. Define the (x_n) in n-normed space $(X, \|., ..., .\|)$ by*

$$x_k = \begin{cases} (0, ..., k) & , k = i^2, i \in \mathbb{N} \\ (0, ..., 0) & , \textit{otherwise.} \end{cases}$$

and let $x = (0, ..., 0)$ and . Then for every $\varepsilon > 0$ and $x_1, x_2, ..., x_{n-1} \in X$

$$\{k \in \mathbb{N} : \|x_k - x, x_1, x_2, ..., x_{n-1}\| \geq \varepsilon\} \subset \left\{1, 4, 9, 16, ..., k^2, ...\right\}.$$

We have that $\delta(\{k \in \mathbb{N} : \|x_k - x, x_1, x_2, ..., x_{n-1}\| \geq \varepsilon\}) = 0$, for every $\varepsilon > 0$ and $z \in X$. This implies that $\mathcal{I}\text{-}\lim_{k\to\infty} \|x_k, x_1, x_2, ..., x_{n-1}\| = \|x, x_1, x_2, ..., x_{n-1}\|$. But, the sequence (x_k) is not convergent to x.

We next provide a proof of the fact that $\mathcal{I}$-limit operation for sequence in n-normed space $(X, \|., ..., .\|)$ is linear with respect to summation and scalar multiplication.

Theorem 1. *Let $\mathcal{I}$ be an admissible ideal. For each $x_1, x_2, ..., x_{n-1}$ in X,*

(i) If $\mathcal{I}\text{-}\lim_{k\to\infty} \|x_k - x, x_1, x_2, ..., x_{n-1}\| = 0$ and $\mathcal{I}\text{-}\lim_{k\to\infty} \|y_k - y, x_1, x_2, ..., x_{n-1}\| = 0$ then $\mathcal{I}\text{-}\lim_{k\to\infty} \|(x_k + y_k) - (x + y), x_1, x_2, ..., x_{n-1}\| = 0$;

(ii) $\mathcal{I}\text{-}\lim_{k\to\infty} \|a(x_k - x), x_1, x_2, ..., x_{n-1}\| = 0$, $a \in \mathbb{R}$;

Proof. (**i**) Let $\varepsilon > 0$ be given. Then set $A_1, A_2 \in \mathcal{I}$ as follows:

$$A_1 = A_1(\varepsilon) := \left\{k \in \mathbb{N} : \|x_k - x, x_1, x_2, ..., x_{n-1}\| \geq \frac{\varepsilon}{2}\right\}$$

and

$$A_2 = A_2(\varepsilon) := \left\{k \in \mathbb{N} : \|y_n - y, x_1, x_2, ..., x_{n-1}\| \geq \frac{\varepsilon}{2}\right\}$$

for each $x_1, x_2, ..., x_{n-1}$ in X. Let

$$A = A(\varepsilon) := \{k \in \mathbb{N} : \|(x_k + y_k) - (x + y), x_1, x_2, ..., x_{n-1}\| \geq \varepsilon\}.$$

Then the inclusion $A \subset A_1 \cup A_2$ holds and the statement follows.

(ii) Let $\mathcal{I}\text{-}\lim_{k\to\infty} \|x_k - x, x_1, x_2, ..., x_{n-1}\| = 0$, $a \in \mathbb{R}$ and $a \neq 0$. Then

$$\left\{k \in \mathbb{N} : \|x_k - k, x_1, x_2, ..., x_{n-1}\| \geq \frac{\varepsilon}{|a|}\right\} \in \mathcal{I}.$$

Then by definition 2.1, we have

$$\begin{aligned} & \{k \in \mathbb{N} : \|ax_k - ax, x_1, x_2, ..., x_{n-1}\| \geq \varepsilon\} \\ = & \{k \in \mathbb{N} : |a| \|x_k - x, x_1, x_2, ..., x_{n-1}\| \geq \varepsilon\} \\ = & \left\{k \in \mathbb{N} : \|x_k - k, x_1, x_2, ..., x_{n-1}\| \geq \frac{\varepsilon}{|a|}\right\}. \end{aligned}$$

Hence, the right hand side of above equality belongs to $\mathcal{I}$. Hence,

$$\mathcal{I} - \lim_{k\to\infty} \|a(x_k - x), x_1, x_2, ..., x_{n-1}\| = 0,$$

for every $x_1, x_2, ..., x_{n-1}$ in X. □

Recall that we assume X to have dimension d, where $n \leq d < \infty$, unless otherwise stated. Let $a = \{a_1, .., a_n\}$ to be a linearly independent set in X. With respect to $\{a_1, .., a_n\}$, if we define the following function $\|., ..., .\|_\infty$ on X^{n-1}

$$\|x_1, ..., x_{n-1}\|_\infty := \max\{\|x_1, ..., x_{n-1}, a_i\| : i = 1, ..., n\}.$$

then the function $\|., ..., .\|_\infty$ defines an $(n-1)$ norm on X [**8**].

LEMMA 2. *Let $\mathcal{I}$ be an admissible ideal. A sequence (x_k) in X is $\mathcal{I}$-convergent to x in X the n-norm if and only if $\mathcal{I}\text{-}\lim_{k\to\infty} \|x_1, x_2, ..., x_{n-2}, x_k - x, a_i\| = 0$ for every $i = 1, ..., n$.*

PROOF. $\mathcal{I}$-convergent to x in X the n-norm then $\mathcal{I}\text{-}\lim_{k\to\infty} \|x_k - x, x_1, x_2, ..., x_{n-1}\| = 0$ for every $x_1, x_2, ..., x_{n-1} \in X$ and $i = 1, 2, ..., n$. Every $z \in X$ can be written as $z = \alpha_1 x_1 + ... + \alpha_n u_n$ for some $\alpha_1, ..., \alpha_n \in \mathbb{R}$. Using the triangle inequality we have

$$\begin{aligned} \|x_1, x_2, ..., x_{n-2}, x_k - x, z\| \leq & |\alpha_1| \|x_1, x_2, ..., x_{n-2}, x_k - x, u_1\| + ... \\ & + |a_n| \|x_1, x_2, ..., x_{n-2}, x_k - x, u_n\| \end{aligned} \tag{2.1}$$

for all $k \in \mathbb{N}$.

If $A(\varepsilon) = \{k \in \mathbb{N} : \|x_1, x_2, ..., x_{n-2}, x_k - x, u_i\| \geq \varepsilon\} \in \mathcal{I}$ for every $\varepsilon > 0$ and $i = 1, ..., n$. From the above inequality we have

$$\{k : \|x_1, x_2, ..., x_{n-2}, x_k - x, z\| \geq \varepsilon\} \subseteq \{k : |\alpha_1| \|x_1, x_2, ..., x_{n-2}, x_k - x, u_1\| \geq \varepsilon\} \cup ... \cup \{n : |\alpha_n| \|x_1, x_2, ..., x_{n-2}, x_k - x, u_n\| \geq \varepsilon\}.$$

Since the right hand side of the above inclusion belongs to ideal, so does the left hand side. □

From Lemma 2.3 and norm $\|., ..., .\|_\infty$, we have:

LEMMA 3. *Let $\mathcal{I}$ be an admissible ideal. A sequence (x_n) in X is $\mathcal{I}$-convergent to x in X in the n-norm if and only if $\mathcal{I}\text{-}\lim_{k\to\infty} \|x_1, x_2, ..., x_{n-2,}x_k - x\|_\infty = 0$.*

Using open balls $B_u(x, \varepsilon)$, we have the following.

LEMMA 4. *Let $\mathcal{I}$ be an admissible ideal. A sequence (x_k) in X is $\mathcal{I}$-convergent to x in X in the n-norm if and only if $A(\varepsilon) = \{k \in \mathbb{N} : x_k \notin B_u(x, \varepsilon)\}$ belongs to ideal.*

Now we introduce the concepts $\mathcal{I}$-Cauchy sequences in n-normed spaces X.

DEFINITION 5. *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a nontrivial ideal in $\mathbb{N}$. The sequence (x_n) of X is said to be $\mathcal{I}$-Cauchy sequence in X, if for each $\varepsilon > 0$ and $z \in X$ there exists a number $N = N(\varepsilon, z)$ such that*

$$\left\{k \in \mathbb{N} : \left\|x_k - x_{N(\varepsilon,z)}, x_1, x_2, ..., x_{n-1}\right\| \geq \varepsilon\right\} \in \mathcal{I}$$

where $z = x_1, x_2, ..., x_{n-1}$.

Further we will give some examples of ideals and corresponding $\mathcal{I}$-convergences.

(I) Let $\mathcal{I}_f$ be the family of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}_f$ is an admissible ideal in $\mathbb{N}$ and $\mathcal{I}_f$ Cauchy sequence coincides with usual Cauchy sequence in [**18**] .

(II) Put $\mathcal{I}_\delta = \{A \subset \mathbb{N} : \delta(A) = 0\}$. Then $\mathcal{I}_\delta$ is an admissible ideal in $\mathbb{N}$ and $\mathcal{I}_\delta$ Cauchy sequence coincides with the statistical Cauchy sequence in [**9**].

Now we give a similar result as in [**8**] (see Lemma 1.2).

LEMMA 5. *In any n-normed space $(X, \|., ..., .\|)$, any $\mathcal{I}$-Cauchy sequences is $\mathcal{I}$-convergent if and only if any $\mathcal{I}$-Cauchy sequence with respect to $\|., ..., .\|_\infty$ is $\mathcal{I}-$convergent.*

PROOF. From Lemma 2.2, $\mathcal{I}$-convergence in the $n-$norm is equivalent to that in the $\|.,...,.\|_\infty$ norm. That is,

$$\begin{aligned}\mathcal{I}\text{-}\lim_{k\to\infty}\|x_1,x_2,...,x_{n-2},x_{n-1},x_k-x\| &= 0, \forall x_1,x_2,...,x_{n-2},x_{n-1}\in X\\ &\Leftrightarrow \mathcal{I}\text{-}\lim_{k\to\infty}\|x_1,x_2,...,x_{n-2},x_k-x\|_\infty .\end{aligned}$$

It sufficient to show that (x_k) is $\mathcal{I}$-Cauchy sequence with respect to the n-norm iff it is $\mathcal{I}$-Cauchy sequence with respect to the norm $\|.,...,.\|_\infty$. But it can be done easily very similar to that in Lemma 1.2 with only mild changes. □

Let X be real inner product space of dimension $d\geq n$ then

$$\|x_1,...,x_n\|_S = \begin{vmatrix} <x_1,x_1> & \cdots & <x_1,x_n> \\ \vdots & \ddots & \vdots \\ <x_n,x_1> & \cdots & <x_n,x_n>\end{vmatrix}^{\frac{1}{2}}$$

where $<.,.>$ denotes the inner product on X. On the other hand, Let $\{e_1,...e_n\}$ be an orthonormal set in X then

$$\|x_1,...,x_n\|_\infty := \max\left\{\|x_1,...,x_{n-1},e_i\| : i=1,...,n\right\}$$

defines an $(n-1)$ norm on X.

Let $(X,\|.,...,.\|)$ be any n-normed spaces and $S(n-X)$ denotes X-valued sequences spaces. Clearly $S(n-X)$ is a linear space under addition and scalar multiplication.

DEFINITION 6. *We define the new sequences space as follows:*

$$l(n-p) = \left\{x\in S(n-X) : \sum_k \|x_k,x_1,x_2,...,x_{n-1}\|^{p_k} < \infty\right\}$$

for each $x_1,x_2,...,x_{n-1}$ *in* X.

LEMMA 6. $l(n-p)$ *sequences space is a linear space.*

PROOF. Let $p_k>0$, $(\forall k)$, $H=\sup\ p_k$ and $a_k,b_k\in\mathbb{C}$ (complex numbers). Then

$$|a_k+b_k|^{p_k}\leq C\left\{|a_k|^{p_k}+|b_k|^{pk}\right\},\quad C=\max\left\{1,2^{H-1}\right\},$$

[**16**]. Hence, if $|\lambda|\leq L$ and $|\mu|\leq M$; L,M integers, $x,y\in l(n-p)$ then we get

$$\begin{aligned}\|\lambda x+\mu y,x_1,x_2,...,x_{n-1}\|^{p_k} &\leq (|\lambda|\,\|x_k,x_1,...,x_{n-1}\|+|\mu|\,\|y_k,x_1,x_2,...,x_{n-1}\|)^{p_k}\\ &\leq (|\lambda|\,\|x_k,x_1,x_2,...,x_{n-1}\|+|\mu|\,\|y_k,x_1,x_2,...,x_{n-1}\|)^{p_k}\\ &\leq (L\,\|x_k,x_1,x_2,...,x_{n-1}\|+M\,\|y_k,x_1,x_2,...,x_{n-1}\|)^{p_k}\\ &\leq C\left\{(L\,\|x_k,x_1,x_2,...,x_{n-1}\|)^{p_k}+(M\,\|y_k,x_1,x_2,...,x_{n-1}\|)^{p_k}\right\}\\ &\leq CL^H\left(\|x_k,x_1,x_2,...,x_{n-1}\|\right)^{p_k}+CM^H\left(\|y_k,x_1,x_2,...,x_{n-1}\|\right)^{p_k}.\end{aligned}$$

Taking sum over k desired result is obtained. □

Definition 7. *Let* $t_k = \sum_{i=1}^{k} (\|x_i, x_1, x_2, ..., x_{n-1}\|)^{p_i}$ *and* $\mathcal{I}$ *be an admissible ideal. Then we define the new sequences space as follows:*

$$l^{\mathcal{I}}(n-p) = \{x \in S(n-X) : \{n \in \mathbb{N} : \|t_k - t, t_1, ..., t_{n-1}\| \geq \varepsilon\} \in \mathcal{I}\}.$$

Theorem 2. *Let* $\mathcal{I}$ *an admissible ideal.* $l^{\mathcal{I}}(n-p)$ *sequences space is a linear space.*

Proof. Using properties of ideal and partial sums of sequences the proof can easily be done similar the above Lemma 2.7. □

Theorem 3. $l(n-p)$ *space is a paranormed spaces with the paranorm defined by* $g : l(n-p) \to \mathbb{R}$,

$$g(x) = \left(\sum_k \|x_k, x_1, x_2, ..., x_{n-1}\|^{p_k} \right)^{\frac{1}{M}}$$

,where $0 < p_k \leq \sup p_k = H,\ M = \max(1, H)$.

Proof. (i) $g(\theta) = \left(\sum_k \|\theta_k, x_1, x_2, ..., x_{n-1}\|^{p_k} \right)^{\frac{1}{M}} = 0.$

(ii) $g(-x) = \left(\sum_k \|-x_k, x_1, x_2, ..., x_{n-1}\|^{p_k} \right)^{\frac{1}{M}}$

$= \left(\sum_k |-1| \|x_k, x_1, x_2, ..., x_{n-1}\|^{p_k} \right)^{\frac{1}{M}} = g(x)$

(iii)

$$\begin{aligned}
g(x+y) &= \left(\sum_k \|x_k + y_k, x_1, x_2, ..., x_{n-1}\|^{p_k} \right)^{\frac{1}{M}} \\
&\leq \left(\sum_k (\|x_k, x_1, x_2, ..., x_{n-1}\| + \| y_k, x_1, x_2, ..., x_{n-1}\|)^{p_k} \right)^{\frac{1}{M}} \\
&\leq \left(\sum_k (\|x_k, x_1, x_2, ..., x_{n-1}\| + \| y_k, x_1, x_2, ..., x_{n-1}\|)^{\frac{p_k}{M} M} \right)^{\frac{1}{M}} \\
&\leq \left(\sum_k \left(\left\|x_k, x_1, x_2, ..., x_{n-1}\right\|^{\frac{p_k}{M}} + \left\| y_k, x_1, x_2, ..., x_{n-1}\right\|^{\frac{p_k}{M}} \right)^M \right)^{\frac{1}{M}} \\
&\leq \left(\sum_k \left(\|x_k, x_1, x_2, ..., x_{n-1}\|^{\frac{p_k}{M}} \right)^M \right)^{\frac{1}{M}} + \left(\sum_k \left(\|y_k, x_1, x_2, ..., x_{n-1}\|^{\frac{p_k}{M}} \right)^M \right)^{\frac{1}{M}} \\
&= g(x) + g(y).
\end{aligned}$$

(iv) Now let $\lambda^n \to \lambda$ and $g(x^n - x) \to 0$ $(n \to \infty)$. We have

$$\begin{aligned} g(\lambda^n x^n - \lambda x) &= \left(\sum_k \| \lambda^n x_k^n - \lambda x, z_1, z_2, ..., z_{n-1} \|^{p_k} \right)^{\frac{1}{M}} \\ &\leq |\lambda|^{n\frac{H}{M}} \left(\sum_k \| x_k^n - x_k, z_1, z_2, ..., z_{n-1} \|^{p_k} \right)^{\frac{1}{M}} \\ &\quad + \left(\sum_k |\lambda^n - \lambda| \, \| x_k, z_1, z_2, ..., z_{n-1} \|^{p_k} \right)^{\frac{1}{M}}. \end{aligned}$$

First statement of the above inequality tends to zero because $g(x^n - x) \to 0$ $(n \to \infty)$. Also, since $\lambda^n \to \lambda$ $(n \to \infty)$ using Lemma 2.7 second statement of the above inequality tends to zero as well. □

THEOREM 4. *If $(X, \|., ..., .\|)$ is finite dimensional $n-$Banach spaces then $(l(n-p), g)$ is complete.*

PROOF. Let (x^n) be a Cauchy sequence in $(l(n-p), g)$. Then for each $\varepsilon > 0$ there exists $N_0 \in \mathbb{N}$ such that for each $m, n > N_0$ we have

$$g(x^n - x^m) = \left(\sum_k \| x_k^n - x_k^m, x_1, x_2, ..., x_{n-1} \|^{p_k} \right)^{\frac{1}{M}} < \varepsilon$$

which implies $\left(\| x^n - x^m, x_1, x_2, ..., x_{n-1} \|^{p_k} \right)^{\frac{1}{M}} < \varepsilon$, for each k. So, (x^n) is a Cauchy sequence in $(X, \|., ..., .\|)$ and since $(X, \|., ..., .\|)$ is n-Banach space there exists an x in X such that $\| x_k^n - x_k, x_1, x_2, ..., x_{n-1} \| \to 0$ $(n \to \infty)$ and this completes the proof. □

THEOREM 5. *If $(X, \|., ..., .\|)$ be any standard n-normed space then*

$$l(n-p)_{\|.,...,.\|_\infty} \equiv l(n-p)_{\|.,...,.\|_{(n-1)S}},$$

that is, $x \in l(n-p)_{\|.,...,.\|_\infty} \Leftrightarrow x \in l(n-p)_{\|.,...,.\|_{(n-1)S}}$

PROOF. From fact 2.3 in [**8**] we have

$$\| x_k, x_1, x_2, ..., x_{n-2} \|_\infty \leq \| x_k, x_1, x_2, ..., x_{n-2} \|_S \leq \sqrt{n} \, \| x_k, x_1, x_2, ..., x_{n-2} \|_\infty$$

for all $z_1, z_2, ..., z_{n-1}$ in X. So we get

$$\begin{aligned} \sum_k \| x_k, x_1, x_2, ..., x_{n-2} \|_\infty^{p_k} &\leq \sum_k \| x_k, x_1, x_2, ..., x_{n-2} \|_S^{p_k} \\ &\leq \sum_k \left[\sqrt{n} \, \| x_k, x_1, x_2, ..., x_{n-2} \|_\infty \right]^{p_k} \\ &\leq n^{\frac{H}{2}} \sum_k \| x_k, x_1, x_2, ..., x_{n-2} \|_\infty^{p_k} \end{aligned}$$

as required. □

THEOREM 6. *$u \in l_\infty \Rightarrow ux \in l(n-p)$, where l_∞ is bounded sequences spaces.*

PROOF. Let $u = (u_k) \in l_\infty$. Then there exists a $A > 0$ such that $|u_k| \leq A$ for each k. We want to show $(u_k x_k) \in l(n-p)$. But

$$\begin{aligned}\sum_k \|u_k x_k, x_1, x_2, ..., x_{n-1}\|^{p_k} &= \sum_k [|u_k| \, \|x_k, x_1, x_2, ..., x_{n-1}\|]^{p_k} \\ &\leq A^H \sum_k \|x_k, x_1, x_2, ..., x_{n-1}\|^{p_k} \\ &< \infty\end{aligned}$$

and this completes the proof. □

Now we give some generalizations of subjects given in [**14**].

DEFINITION 8. *Let $A = (a_{m,k})$ be a non-negative matrix. Define the new sequences space as follows:*

$$w_0(n-p) = \left\{ x \in S(n-X) : \lim_{m\to\infty} \sum_k \|a_{m,k} x_k, x_1, x_2, ..., x_{n-1}\|^{p_k} = 0 \right\}$$

for each $x_1, x_2, ..., x_{n-1}$ in X. If $x - te \in w_0(n-p)$ then we say x is $w_0(n-p)$ summable to t, where, $e = (1, 1, ...)$.

THEOREM 7. *$w_0(n-p)$ is linear.*

PROOF. It can be done very similar to the proof of linearity of $l(n-p)$ □

THEOREM 8. *$w_0(n-p)$ is paranormed space by*

$$g(x) = \sup_m \left(\sum_k \|a_{m,k} x_k, x_1, x_2, ..., x_{n-1}\|^{p_k} \right)^{\frac{1}{M}}$$

PROOF. Again it is very similar to above one we omit it. □

THEOREM 9. *If $A = (a_{m,k})$ is the matrix of Cesaro means of order 1 then $l(n-p) \subseteq w_0(n-p)$.*

PROOF. If $A = (a_{m,k})$ is the matrix of Cesaro means of order 1 then

$$\begin{aligned}A_m(x) &= \sum_{k=1}^{\infty} \|a_{m,k} x_k, x_1, x_2, ..., x_{n-1}\|^{p_k} \\ &\leq \frac{1}{m} \sum_{k=1}^{\infty} \|x_k, x_1, x_2, ..., x_{n-1}\|^{p_k}\end{aligned}$$

So, if $x \in l(n-p)$ then there exists $M > 0$ such that

$$\sum_{k=1}^{\infty} \|x_k, x_1, x_2, ..., x_{n-1}\|^{p_k} = M > 0.$$

Hence $0 \leq \lim_{m\to\infty} A_m(x) \leq \frac{M}{m} = 0$. This means $x \in w_0(n-p)$. □

More generally we have the following result.

THEOREM 10. *If $A = (a_{m,k})$ is any regular matrix then $l(n-p) \subseteq w_0(n-p)$.*

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