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Fekete-Szegö Inequality for a Subclass of Bi-univalent Functions by Applying Sălăgean q-Differential Operator

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Abstract Throughout this study, we propose a new subclass of bi-univalent functions by applying the Sălăgean q-differential operator and denoted as $\mathcal{L}\Sigma_q^k(\lambda,\phi)$. Additionally, we acquired the values of the initial coefficients $|a_2|$ and $|a_3|$ for functions $f \in \mathcal{L}\Sigma_q^k(\lambda,\phi)$ which yield to this study's preliminary result. Subsequently, the preliminary result was applied to obtain the upper bound of Fekete-Szegö inequality, $|a_3 - \rho a_2^2|$, for functions $f \in \mathcal{L}\Sigma_q^k(\lambda,\phi)$.

Keywords: Fekete-Szegö Inequality, Bi-univalent Functions, Sălăgean q-Differential Operator.

Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Furthermore, let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) which are univalent in $\mathcal{U}$.

For the two functions f and g, that are analytic in $\mathcal{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathcal{U}$, and write $f \prec g$ or $f(z) \prec g(z)$, $z \in \mathcal{U}$, if there exists a Schwarz function $w(z)$, analytic in $\mathcal{U}$ with $w(0) = 0$ and $|w(z)| < 1$, $z \in \mathcal{U}$, such that $f(z) = g\big(w(z)\big)$, $z \in \mathcal{U}$. In particular, if the function g is univalent in $\mathcal{U}$, the subordination is equivalent to $f(0) = g(0)$ and $f(\mathcal{U}) \subset g(\mathcal{U})$. (see [23])

Apart from that, function f which belongs to $\mathcal{S}$ has an inverse f^{-1} that can be written as $f^{-1}\big(f(z)\big) = z$, $(z \in \mathcal{U})$ and $f\big(f^{-1}(w)\big) = w$ $\left(|w| < r_0(f), r_0(f) \geq \frac{1}{4}\right)$. A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathcal{U}$ if both f and f^{-1} are univalent in $\mathcal{U}$.

Throughout this study, let Σ denote the class of bi-univalent functions defined in $\mathcal{U}$. Since $f \in \Sigma$ has the Maclaurin series given by (1.1), its inverse $g = f^{-1}$ can be shown as the expansion of

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 + \cdots. \tag{1.2}$$

Currently, various subclasses of bi-univalent functions have been introduced by mathematicians and the study of coefficient problems, especially the Hankel determinant, is still actively studied. (see [9])

Noonan and Thomas [19] gave a definition for the mth Hankel determinant of f for integers $n \geq 1$ and $m \geq 1$ as

$$H_m(n) = \begin{vmatrix} a_n & \cdots & a_{n+m-1} \\ \vdots & \cdots & \vdots \\ a_{n+m-1} & \cdots & a_{n+2m-2} \end{vmatrix}, \quad a_1 = 1.$$

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By considering several values for m and n, the Hankel determinant $H_2(1)$, $H_2(2)$ and $H_2(3)$ will be obtained. There are many results related to the results of $H_2(1)$, $H_2(2)$ and $H_2(3)$ for subclasses of univalent and bi-univalent functions that have been widely explored by mathematicians, such as [4, 5, 8, 9, 13, 14, 15, 18, 24].

Recently, the field of q-calculus has become a research trend among mathematicians. Researchers are interested in conducting research in this field because of its application in various branches of mathematics and physics. The application of q-calculus was initiated by Jackson [12]. He was the first to develop the q-integral and q-derivative in a systematic way.

For a function $f \in \mathcal{A}$ given by (1.1) and $0 < q < 1$, the q-derivative of a function f is defined by Jackson [12]

$$\mathcal{D}_q f(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z} , & for \quad z \neq 0 \\ f'(0) , & for \quad z = 0 \end{cases} \tag{1.3}$$

and $\mathcal{D}_q^2 f(z) = \mathcal{D}_q\left(\mathcal{D}_q f(z)\right)$. From (1.3), Jackson [12] has shown that

$$\mathcal{D}_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \tag{1.4}$$

where $[n]_q$ in (1.4) can be calculated by the formulae

$$[n]_q = \frac{1-q^n}{1-q}. \tag{1.5}$$

If $q \to 1^-$ in the formulae (1.5) then $[n]_q \to n$.

Besides that, Sălăgean [22] has proposed the following Sălăgean differential operator for $f(z) \in \mathcal{A}$ as follows :

$$\mathcal{D}^0 f(z) = f(z),$$
$$\mathcal{D}^1 f(z) = \mathcal{D}f(z) = zf'(z),$$
$$\mathcal{D}^k f(z) = \mathcal{D}\left(\mathcal{D}^{k-1} f(z)\right) \quad (k \in \mathbb{N} = 1,2,3,\cdots).$$

By substituting $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, the kth order of differential operator will be

$$\mathcal{D}^k f(z) = z + \sum_{n=2}^{\infty} n^k a_n z^n, \quad k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \tag{1.6}$$

Numerous authors have been exploring the Sălăgean differential operator in the past few years, among them are as [3, 7, 11].

Further, Govindaraj and Sivasubramanian [6] had generalized (1.6) and defined the Sălăgean q-differential operator for $f \in \mathcal{A}$ as given below:

$$\mathcal{D}_q^0 f(z) = f(z),$$
$$\mathcal{D}_q^1 f(z) = z\mathcal{D}_q f(z),$$
$$\mathcal{D}_q^k f(z) = z\mathcal{D}_q^k\left(\mathcal{D}_q^{k-1} f(z)\right),$$
$$\mathcal{D}_q^k f(z) = z + \textstyle\sum_{n=2}^{\infty} [n]_q^k a_n z^n \quad (k \in \mathbb{N}_0, z \in \mathcal{U}). \tag{1.7}$$

The study associated with the Sălăgean q-differential operator had been comprehensively studied by such researchers [2, 10, 16, 17].

Several authors had investigated the Fekete-Szegö functional $H_2(1)$ for various subclasses of bi-univalent functions associated with the Sălăgean q-differential operator (see [20, 25, 26]). Motivated by that investigation, using the Sălăgean q-differential operator given by (1.7) and the principle of subordination, for functions g of the form (1.2), we define

$$\mathcal{D}_q^k g(w) = w - a_2[2]_q^k w^2 + (2a_2^2 - a_3)[3]_q^k w^3 + \cdots, \tag{1.8}$$

we also introduce a new subclass of Σ which is denoted by $\mathcal{L}\Sigma_q^k(\lambda,\phi)$. The target of this study is to determine the upper bound of Fekete-Szegö functional $H_2(1) = |a_3 - \rho a_2^2|$, for the function f belongs to $\mathcal{L}\Sigma_q^k(\lambda,\phi)$. We begin with the following definition first.

Let $\phi(0) = 1, \phi'(0) > 0$, be an analytic function in $\mathcal{U}$ with positive real part, which is symmetrical with respect to the real axis. The function has a series expansion of the form

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \quad (B_1 > 0). \tag{1.9}$$

Now, we present a new subclass of Σ as the following.

Definition 1.1. For $0 \leq \lambda < 1$, a function $f \in \Sigma$ of the form (1.1) is said to be in the class $\mathcal{L}\Sigma_q^k(\lambda,\phi)$ if the following subordination hold

$$\frac{\mathcal{D}_q^{k+1} f(z)}{(1-\lambda)\mathcal{D}_q^k f(z) + \lambda \mathcal{D}_q^{k+1} f(z)} \prec \phi(z),$$

and

$$\frac{\mathcal{D}_q^{k+1} g(w)}{(1-\lambda)\mathcal{D}_q^k g(w) + \lambda \mathcal{D}_q^{k+1} g(w)} \prec \phi(w),$$

where $\mathcal{D}_q^k g(w)$ is given by (1.8) and (1.7).

Remark 1.1. For $0 \leq \lambda < 1$ and $k = 0$, a function $f \in \Sigma$ of the form (1.1) is said to be in the class $\mathcal{L}\Sigma_q^0(\lambda,\phi)$ if the following subordination hold

$$\frac{z\mathcal{D}_q f(z)}{(1-\lambda)f(z) + \lambda\left(z\mathcal{D}_q f(z)\right)} \prec \phi(z),$$

and

$$\frac{w\mathcal{D}_q g(w)}{(1-\lambda)g(w) + \lambda\left(w\mathcal{D}_q g(w)\right)} \prec \phi(w),$$

where $z, w \in \mathcal{U}$ and $\mathcal{D}_q^k g(w)$ is given by (1.8).

To obtain the upper bound of $|a_3 - \rho a_2^2|$ for $f \in \mathcal{L}\Sigma_q^k(\lambda,\phi)$, we need the coefficients $|a_2|$ and $|a_3|$, which will be discussed in the following section.

Lemma 1.1 ([21]) If a function $p \in \mathcal{P}$ is given by

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \qquad (z \in \mathcal{U}),$$

then

$$|p_i| \leq 2 \qquad (i \in \mathbb{N}),$$

where $\mathcal{P}$ is the family of all functions p, analytic in $z \in \mathcal{U}$, for which

$$p(0) = 1 \qquad and \qquad Re(p(z)) > 0 \qquad (z \in \mathcal{U}).$$

Main Results

Next, we state our main result. Before that, we get the values for the initial coefficients a_2 and a_3.

Lemma 2.1 Let f given by (1.1) be in the class $\mathcal{L}\Sigma_q^k(\lambda,\phi)$. Then

$$|a_2| \leq \frac{\beta_1\sqrt{\beta_1}}{\sqrt{\left|\left((\lambda^2-1)[2]_q^{2k}+2(1-\lambda)[3]_q^k\right)\beta_1^2+(1-\lambda)^2[2]_q^{2k}(\beta_1-\beta_2)\right|}}$$

and

$$|a_3| \leq \frac{\beta_1}{2(1-\lambda)[3]_q^k}+\left(\frac{\beta_1}{(1-\lambda)[2]_q^k}\right)^2$$

where $0 \leq \lambda < 1$.

Proof Let $f \in \mathcal{L}\Sigma_q^k(\lambda,\phi)$ and $g = f^{-1}$. Then there are analytic functions $u, v : \mathcal{U} \to \mathcal{U}$, with $u(0) = 0 = v(0)$, satisfying

$$\frac{\mathcal{D}_q^{k+1}f(z)}{(1-\lambda)\mathcal{D}_q^k f(z)+\lambda\mathcal{D}_q^{k+1}f(z)} = \phi(u(z)) \tag{2.1}$$

and

$$\frac{\mathcal{D}_q^{k+1}g(w)}{(1-\lambda)\mathcal{D}_q^k g(w)+\lambda\mathcal{D}_q^{k+1}g(w)} = \phi(v(w)). \tag{2.2}$$

Let us first work on the left-hand side of the above equations. From (2.1) and (1.7), we have

$$\begin{aligned}\mathcal{D}_q^k f(z) &= z+[2]_q^k a_2 z^2+[3]_q^k a_3 z^3+\cdots\\ \mathcal{D}_q^{k+1} f(z) &= z+2[2]_q^k a_2 z^2+3[3]_q^k a_3 z^3+\cdots\ .\end{aligned} \tag{2.3}$$

Then,

$$(1-\lambda)\mathcal{D}_q^k f(z)+\lambda\mathcal{D}_q^{k+1}f(z) = z+(1+\lambda)[2]_q^k a_2 z^2+(1+2\lambda)[3]_q^k a_3 z^3+\cdots. \tag{2.4}$$

Therefore, by dividing equation (2.3) with equation (2.4), we obtain

$$\begin{aligned}&\frac{\mathcal{D}_q^{k+1}f(z)}{(1-\lambda)\mathcal{D}_q^k f(z)+\lambda\mathcal{D}_q^{k+1}f(z)}\\ &\qquad = 1+(1-\lambda)[2]_q^k a_2 z+\left[2(1-\lambda)[3]_q^k a_3-(1-\lambda^2)[2]_q^{2k}a_2^2\right]z^2+\cdots.\end{aligned} \tag{2.5}$$

For (2.2), let

$$\begin{aligned}\mathcal{D}_q^k g(w) &= w-[2]_q^k a_2 w^2+[3]_q^k(2a_2^2-a_3)w^3+\cdots\\ \mathcal{D}_q^{k+1} g(w) &= w\left(1-2[2]_q^k a_2 w+3[3]_q^k(2a_2^2-a_3)w^2+\cdots\right)\\ &= w-2[2]_q^k a_2 w^2+3[3]_q^k(2a_2^2-a_3)w^3+\cdots.\end{aligned} \tag{2.6}$$

Then,

$$\begin{aligned}&(1-\lambda)\mathcal{D}_q^k g(w)+\lambda\mathcal{D}_q^{k+1}g(w)\\ &\qquad = w-(1+\lambda)[2]_q^k a_2 w^2+(1+2\lambda)[3]_q^k(2a_2^2-a_3)w^3+\cdots.\end{aligned} \tag{2.7}$$

Therefore, by dividing equation (2.6) with equation (2.7), we have

$$\begin{aligned}&\frac{\mathcal{D}_q^{k+1}g(w)}{(1-\lambda)\mathcal{D}_q^k g(w)+\lambda\mathcal{D}_q^{k+1}g(w)}\\ &\qquad = 1+(\lambda-1)[2]_q^k a_2 w+\left[(\lambda^2-1)[2]_q^{2k}a_2^2+2(1-\lambda)[3]_q^k(2a_2^2-a_3)\right]w^2+\cdots.\end{aligned} \tag{2.8}$$

Now, for the right-hand side of equation (2.1) and (2.2), we define the functions $m(z)$ and $n(z)$ by

$$m(z) = \frac{1+u(z)}{1-u(z)} = 1+m_1 z+m_2 z^2+\cdots, \tag{2.9}$$

and

$$n(z)=\frac{1+v(z)}{1-v(z)}=1+n_1z+n_2z^2+\cdots. \tag{2.10}$$

or, equivalently, from (2.9) and (2.10), we obtain

$$u(z)=\frac{1}{2}\left[m_1z+\left(m_2-\frac{m_1^2}{2}\right)z^2+\cdots\right] \tag{2.11}$$

and

$$v(z)=\frac{1}{2}\left[n_1z+\left(n_2-\frac{n_1^2}{2}\right)z^2+\ldots\right]. \tag{2.12}$$

Then, $m(z)$ and $n(z)$ are analytic in $\mathcal{U}$ with $m(0)=1=n(0)$. Since $u,v:\mathcal{U}\to\mathcal{U}$, the functions $m(z)$ and $n(z)$ have a positive real part in $\mathcal{U}$, $|m_i|\leq 2$ and $|n_i|\leq 2$.
Substituting equation (2.11) to equation (1.9), we obtain

$$\phi(u(z))=1+\frac{1}{2}\beta_1m_1z+\left[\frac{1}{2}\beta_1\left(m_2-\frac{m_1^2}{2}\right)+\frac{1}{4}\beta_2m_1^2\right]z^2+\cdots. \tag{2.13}$$

Substituting equation (2.12) into equation (1.9), we acquire

$$\phi(v(w))=1+\frac{1}{2}\beta_1n_1w+\left[\frac{1}{2}\beta_1\left(n_2-\frac{n_1^2}{2}\right)+\frac{1}{4}\beta_2n_1^2\right]w^2+\cdots. \tag{2.14}$$

Substituting equation (2.5) and (2.13) into (2.1), we have

$$\begin{aligned}1+(1-\lambda)[2]_q^ka_2z+\left[2(1-\lambda)[3]_q^ka_3-(1-\lambda^2)[2]_q^{2k}a_2^2\right]z^2+\cdots\\=1+\frac{1}{2}\beta_1m_1z+\left[\frac{1}{2}\beta_1\left(m_2-\frac{m_1^2}{2}\right)+\frac{1}{4}\beta_2m_1^2\right]z^2+\cdots.\end{aligned} \tag{2.15}$$

Comparing the coefficients of z and z^2 of both sides of equation (2.15), we get

$$z:\qquad (1-\lambda)[2]_q^ka_2=\frac{1}{2}\beta_1m_1 \tag{2.16}$$

$$z^2:\qquad 2(1-\lambda)[3]_q^ka_3-(1-\lambda^2)[2]_q^{2k}a_2^2=\frac{1}{2}\beta_1\left(m_2-\frac{m_1^2}{2}\right)+\frac{1}{4}\beta_2m_1^2 \tag{2.17}$$

Substitute (2.8) and (2.14) into (2.2),

$$\begin{aligned}1+(\lambda-1)[2]_q^ka_2w+\left[(\lambda^2-1)[2]_q^ka_2^2+2(1-\lambda)[3]_q^k(2a_2^2-a_3)\right]w^2\\=1+\frac{1}{2}\beta_1n_1w+\left[\frac{1}{2}\beta_1\left(n_2-\frac{n_1^2}{2}\right)+\frac{1}{4}\beta_2n_1^2\right]w^2+\cdots\end{aligned} \tag{2.18}$$

Comparing the coefficients of w and w^2 of both sides of equation (2.18), we have

$$w:\qquad (\lambda-1)[2]_q^ka_2=\frac{1}{2}\beta_1n_1 \tag{2.19}$$

$$w^2:\qquad (\lambda^2-1)[2]_q^ka_2^2+2(1-\lambda)[3]_q^k(2a_2^2-a_3)=\frac{1}{2}\beta_1\left(n_2-\frac{n_1^2}{2}\right)+\frac{1}{4}\beta_2n_1^2. \tag{2.20}$$

From (2.16) and (2.19), it shows that

$$m_1=-n_1 \tag{2.21}$$

and

$$m_1^2+n_1^2=\frac{8(1-\lambda)^2[2]_q^{2k}a_2^2}{\beta_1^2}. \tag{2.22}$$

From (2.17), (2.20) and (2.22), we acquire

$$a_2^2 = \frac{\beta_1^3(m_2+n_2)}{4\left[\left((\lambda^2-1)[2]_q^{2k}+2(1-\lambda)[3]_q^k\right)\beta_1^2+(1-\lambda)^2[2]_q^{2k}(\beta_1-\beta_2)\right]}. \tag{2.23}$$

As a result, by taking the modulus of both sides of equation (2.23) and applying Lemma 1.1 to the coefficient $|m_2|$ and $|n_2|$, we attain

$$|a_2| \le \frac{\beta_1\sqrt{\beta_1}}{\sqrt{\left|\left((\lambda^2-1)[2]_q^{2k}+2(1-\lambda)[3]_q^k\right)\beta_1^2+(1-\lambda)^2[2]_q^{2k}(\beta_1-\beta_2)\right|}}.$$

By subtracting equation (2.17) from equation (2.20), then using (2.21) and (2.22), we get

$$a_3 = \frac{\beta_1(m_2-n_2)}{8(1-\lambda)[3]_q^k} + \frac{\beta_1^2(m_1^2+n_1^2)}{8(1-\lambda)^2[2]_q^{2k}}. \tag{2.24}$$

By taking the modulus on both sides of equation (2.24) and utilizing Lemma 1.1 once again to the coefficients $|m_1|$, $|m_2|$, $|n_1|$ and $|n_2|$, we obtain

$$|a_3| \le \frac{\beta_1}{2(1-\lambda)[3]_q^k} + \left(\frac{\beta_1}{(1-\lambda)[2]_q^k}\right)^2.$$

Therefore, the proof for Lemma 2.1 is completed.

For $\lambda = 0$ in Lemma 2.1, we have the following result.

Corollary 2.1. Let f given by (1.1) be in the class $\mathcal{L}\Sigma_q^k(0,\phi)$. Then

$$|a_2| \le \frac{\beta_1\sqrt{\beta_1}}{\sqrt{\left|\left(-[2]_q^{2k}+2[3]_q^k\right)\beta_1^2+[2]_q^{2k}(\beta_1-\beta_2)\right|}}$$

and

$$|a_3| \le \frac{\beta_1}{2[3]_q^k} + \left(\frac{\beta_1}{[2]_q^k}\right)^2$$

From Remark 1.1, Lemma 2.1 generates the following corollary.

Corollary 2.2. Let f given by (1.1) be in the class $\mathcal{L}\Sigma_q^0(\lambda,\phi)$. Then

$$|a_2| \le \frac{\beta_1\sqrt{\beta_1}}{\sqrt{\left|\left((\lambda^2-1)[2]_q+2(1-\lambda)[3]_q\right)\beta_1^2+(1-\lambda)^2[2]_q(\beta_1-\beta_2)\right|}}$$

and

$$|a_3| \le \frac{\beta_1}{2(1-\lambda)[3]_q} + \left(\frac{\beta_1}{(1-\lambda)[2]_q}\right)^2.$$

The main result is stated as follows.

Theorem 2.1. Let the function $f(z) \in \mathcal{L}\Sigma_q^k(\lambda,\phi)$ and $\rho \in \mathbb{C}$, then

$$|a_3-\rho a_2^2| \le \begin{cases} \dfrac{B_1}{2(1-\lambda)[3]_q^k}, & 0 \le |\Theta(\rho)| < \dfrac{1}{8(1-\lambda)[3]_q^k}, \\ 4B_1|\Theta(\rho)|, & |\Theta(\rho)| \ge \dfrac{1}{8(1-\lambda)[3]_q^k}, \end{cases}$$

where

$$\Theta(\rho) = \frac{\beta_1^2(1-\rho)}{4\left[\left((\lambda^2-1)[2]_q^{2k}+2(1-\lambda)[3]_q^k\right)\beta_1^2+(1-\lambda)^2[2]_q^{2k}(\beta_1-\beta_2)\right]}.$$

Proof. From (2.24), we know that

$$a_3 = \frac{\beta_1(m_2-n_2)}{8(1-\lambda)[3]_q^k} + a_2^2.$$

Hence,

$$a_3 - \rho a_2^2 = \frac{\beta_1(m_2 - n_2)}{8(1-\lambda)[3]_q^k} + (1-\rho)a_2^2. \tag{2.25}$$

Substituting equation (2.23) to (2.25), we have

$$a_3 - \rho a_2^2 = \beta_1\left[\left(\Theta(\rho) + \frac{1}{8(1-\lambda)[3]_q^k}\right)m_2 + \left(\Theta(\rho) - \frac{1}{8(1-\lambda)[3]_q^k}\right)n_2\right],$$

where

$$\Theta(\rho) = \frac{\beta_1^2(1-\rho)}{4\left[\left((\lambda^2-1)[2]_q^{2k} + 2(1-\lambda)[3]_q^k\right)\beta_1^2 + (1-\lambda)^2[2]_q^{2k}(\beta_1 - \beta_2)\right]}.$$

Since all β_j are real and $\beta_1 > 0$, we have

$$|a_3 - \rho a_2^2| \leq 2\beta_1\left|\left(\Theta(\rho) + \frac{1}{8(1-\lambda)[3]_q^k}\right) + \left(\Theta(\rho) - \frac{1}{8(1-\lambda)[3]_q^k}\right)\right|$$

where

$$|a_3 - \rho a_2^2| \leq \frac{\beta_1}{2(1-\lambda)[3]_q^k} \text{ for } 0 \leq |\Theta(\rho)| < \frac{1}{8(1-\lambda)[3]_q^k}$$

and

$$|a_3 - \rho a_2^2| \leq 4\beta_1|\Theta(\rho)| \text{ for } |\Theta(\rho)| \geq \frac{1}{8(1-\lambda)[3]_q^k}.$$

Therefore, the proof for Theorem 2.1 is completed.

For $\lambda = 0$ in Theorem 2.1, we obtain the following result.

Corollary 2.3. Let the function $f(z) \in \mathcal{L}\Sigma_q^k(0,\phi)$ and $\rho \in \mathbb{C}$, then

$$|a_3 - \rho a_2^2| \leq \begin{cases} \frac{B_1}{2[3]_q^k}, & 0 \leq |\Theta(\rho)| < \frac{1}{8[3]_q^k}, \\ & g \\ 4B_1|\Theta(\rho)|, & |\Theta(\rho)| \geq \frac{1}{8[3]_q^k}, \end{cases}$$

where

$$\Theta(\rho) = \frac{\beta_1^2(1-\rho)}{4\left[\left(-[2]_q^{2k} + 2[3]_q^k\right)\beta_1^2 + [2]_q^{2k}(\beta_1 - \beta_2)\right]}.$$

Conclusions

Throughout this study, a new subclass of bi-univalent functions by applying the Sălăgean q-differential operator had been presented. Along with that, we had also determined the initial coefficients, $|a_2|$ and $|a_3|$ and the upper bound of Fekete-Szegö inequality for function f belongs to the new subclass $\mathcal{L}\Sigma_q^k(\lambda,\phi)$ had been discovered.

Conflicts of Interest

The author(s) declare(s) that there is no conflict of interest regarding the publication of this paper.

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References

[1] Ali, R. M., Lee, S. K., Ravichandran V. & Supramaniam S. (2011). Coefficient estimates for bi-univalent Ma-Minda starlike and convex functions. *Applied Mathematics Letters*, *25*(3), 344-351.

[2] Aouf, M. K., Mostafa, A. O. & Morsy, R. E. E. L. (2020). Coefficient bounds for general class of bi-univalent functions of complex order associated with q- Sălăgean operator and Chebyshev polynomials. *Electronic Journal of Mathematical Analysis and Applications*, *8*(2), 251-260.

[3] Çağlar, M. & Deniz, E. (2017). Initial Coefficients for a subclass of bi-univalent functions defined by Salagean differential operator. *Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics*, *66*(1), 85-91.

[4] Choo, C. P. & Janteng, A. (2013). Estimate on the second hankel functional for a subclass of close-to-convex functions with respect to symmetric points. *International Journal of Mathematical Analysis*, *7*(13-16), 781-788

[5] Frasin, B. A. & Aouf, M. K. (2011). New subclasses of bi-univalent functions. *Appl. Math. Lett.*, *24*, 1569-1573.

[6] Govindaraj, M. & Sivasubramanian, S. (2017). On a class of analytic function related to conic domains involving $q-$calculus. *Analysis Math.*, *43*(3), 475-487.

[7] Halim, S. A. (1992). On a class of analytic functions involving the Salagean Differential Operator. *Tamkang Journal of Mathematics*, *23*(1), 51-58.

[8] Halim, S. A., Janteng, A. & Darus, M. (2006). Classes with negative coefficients and starlike with respect to other points II. *Tamkang Journal of Mathematics*, *37*(4), 345-354.

[9] Huey, K. S., Janteng, A., Janteng, J. & Hern, A. L. P. (2023). Second Hankel determinant of bi-univalent functions. *Malaysian Journal of Fundamental and Applied Sciences*, *19*(2), 269-279.

[10] Hussain, S., Khan, S., Zaighum, M. A. & Darus, M. (2017). Certain subclass of analytic functions related with conic domains and associated with Salagean q-differential operator. *AIMS Mathematics*, *2*(4), 622-634.

[11] Ibrahim, R. W. & Darus M. (2019). Subordination inequalities of a new Salagean-difference operator. *International Journal of Mathematics*, *14*(3), 573-582.

[12] Jackson, F. H. (1908). On q-functions and a certain difference operator. *Transactions of the Royal Society of Edinburgh*, *46*(2), 253-281.

[13] Janteng, A. & Halim, S. A. (2009). A subclass of quasi-convex functions with respect to symmetric points, *Applied Mathematical Sciences*, *3*(12), 551-556.

[14] Li, X. F. & Wang, A. P. (2012). Two new subclasses of bi-univalent functions. *Int. Math. Forum*, *7*, 1495-1504.

[15] Liew, A. P. H., Janteng, A. & Omar, R. (2020). Hankel determinant $H_2(3)$ for certain subclasses of univalent functions. *Mathematics and Statistics*, *8*(5), 566-569.

[16] Murugusundaramoorthy, G. & Vijaya K. (2017). Subclasses of bi-univalent functions defined by Sălăgean type q-difference operator. arXiv:1710.00143v1.

[17] Murugusundaramoorthy, G., Yalçin, S. & Altınkaya, Ş. (2019). Fekete–Szegö inequalities for subclass of bi-univalent functions associated with Sălăgean type q-difference operator. *Afrika Matematika, 30*, 979-987.

[18] Mustafa, N., Murugusundaramoorthy, G. & Janani, T. (2018). Second Hankel determinant for a certain subclass of bi-univalent functions. *Mediterranean Journal of Mathematics*, *15*(3), 1-17.

[19] Noonan, J. W. & Thomas, D. K. (1976). On the second Hankel determinant of areally mean p-valent functions. *Trans. Am. Math. Soc.*, *223*, 337-346.

[20] Orhan, H., Magesh, N., Balaji, V. K. (2016). Fekete–Szegö problem for certain classes of Ma-Minda bi-univalent functions. *Afrika Matematika, 27*(5-6), 889-897.

[21] Pommerenke, C. (1975). U*nivalent functions.* Vandenhoeck and Ruprecht, Göttingen.

[22] Sălăgean, G. S. (1983). Subclasses of univalent functions, *Complex analysis – Proceedings 5th Romanian-Finnish Seminar*, Busharest, *1013*, 362-372.

[23] Srivastava, H. M. & Attiya, A. A. (2004). Some subordination results associated with certain subclass of analytic functions. *Appl. Math. Sci.*, *5*(4), 1-6.

[24] Srivastava, H. M., Mishra, A. K. & Gochhayat, P. (2010). Certain subclasses of analytic and bi-univalent functions. *Applied Mathematics Letters*, *23*(10), 1188-1192.

[25] Tang, H., Srivastava, H. M., Sivasubramanian, S. & Gurusamy, P. (2016). The Fekete–Szegö functional problems for some subclasses of m–fold symmetric bi–univalent functions. *Journal of Mathematical Inequalities*, *10*(4), 1063-1092.

[26] Zaprawa, P. (2014). On the Fekete-Szegö problem for classes of bi-univalent functions. *Bull. Belg. Math. Soc. Simon Stevin*, *21*(1), 169-178.