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RESEARCH ARTICLE

The First Zagreb Index of the Zero Divisor Graph for the Ring of Integers Modulo Power of Primes

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Abstract Let Γ be a simple graph with the set of vertices and edges. The first Zagreb index of a graph is defined as the sum of the degree of each vertex to the power of two. Meanwhile, the zero divisor graph of a ring R, denoted by $\Gamma(R)$, is defined as a graph with its vertex set $Z(R)^*$ contains the nonzero zero divisors in which two distinct vertices u and v are adjacent if $uv = vu = 0$. In this paper, the general formula of the first Zagreb index of the zero divisor graph for the commutative ring of integers modulo p^k, $\mathbb{Z}_{p^k}$ where a prime number p and a positive integer k is determined. A few examples are given to illustrate the main results.

Keywords: Topological index, first Zagreb index, zero divisor graph, graph theory, ring theory.

Introduction

Chemical properties can be explained by topological indices, also known as molecular structure descriptors. Using topological indices, it is possible to establish numerical relationships between the chemical structure of a molecule and its physical properties, chemical reactivity or biological action. In this way, computing the topological indices of a graph obtained from an algebraic structure such as a ring is very interesting to study. A standard graph notation $\Gamma = (V, E)$ can be defined as a pair consisting of a set whose elements are called vertices denoted by V and a set of edges, which is denoted by E can be represented by many different types of indices such as the Wiener index, the Randić index, the Szeged index, the Harary index, the degree-distance index, the Zagreb index and many more. A topological index was first used in chemistry in 1947 by a chemist Harold Wiener to determine the physical properties of paraffins, which were alkanes with different types of topological descriptors for their properties [1].

Due to its close relationship to chemistry, Basak *et al*. [2] studied the nature, and applications of topological indices. Darafsheh in [3] used the group of automorphisms of Γ to determine the Padmakar-Ivan index, the Szeged index, and the Wiener index for various graphs. This is a fast way to find these indices, especially when V or E has a few orbits in the automorphism group of Γ [3]. It was shown by Gutman and Das in [4] that the first Zagreb index is related to several other quantities of interest in chemical graph theory, and some of its general mathematical properties were outlined. The computation of Zagreb eccentricity index is included by Ghorbani and Hosseinzadeh in [5]. Alimon *et al*. [6] generalised the first Zagreb and the second Zagreb indices of the non-commuting graph of the dihedral groups. A study performed by Das *et al*. [7] identified lower and upper bounds on the first Zagreb index can be defined based on the vertices and edges, as well as the maximum and minimum vertex degrees.

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Aykaç *et al*. [8] have developed the Zagreb indices and multiplicative Zagreb indices of $\Gamma(\mathbb{Z}_{p^2} \times \mathbb{Z}_{q^2})$. Meanwhile, the first Zagreb index of the zero divisor type graph is computed based on the degree of all vertices in the graph by Mazlan *et al*. in [9].

There are four distinct kinds of operations that Pattabiraman and Suganya present in [10] for both edge and vertex versions of two connected graphs. The zero divisor graph from commutative rings has been illustrated by Rayer and Jeyaraj in [11]. They have also discussed topological indices for zero divisor graphs based on eccentricity of the vertex. A zero-divisor graph with nonzero zero divisors as its vertices has been proposed and studied by Anderson and Livingston in [12]. This graph is able to best illustrate the properties of a commutative ring's set of zero divisors. Shuker and Rasheed [13] have defined the Maxideal and Minideal zero divisor graphs of the ring $\mathbb{Z}_n$ for any n, to be the zero divisor graph of maximal and minimal ideals of the ring $\mathbb{Z}_n$. The paper of Smith in [14] focused on the graph of zero divisors of the set of integers modulo n, $\mathbb{Z}_n$. Seeta in [15] examined the girth, diameter, and connectivity of the zero divisor graph and studied $\Gamma(R)$ for non-isomorphic rings. In [16], Anderson carried on his work with Weber to investigate $\Gamma(R)$ when R has no identity, and they found all zero divisor graphs with the number of vertices must be less than 14. In their study of zero divisor graphs of finite commutative rings, Singh and Bhat [17] examined their structural properties. In determining zero divisor graphs of a commutative ring R, Coykendall *et al*. [18] studied equivalence classes between zero divisors. Baruah in [19] studied the zero divisor graph of the ring of Gaussian integer and discussed the diameter, number of vertices, and girth of the graphs. Zaid *et al.* [20] computed the two finite rings with zero divisors, which are the ring of two-by-two matrices over integers modulo two and three. Kuppan and Ravi Sankar [21] presented a prime decomposition of a zero divisor graph in a commutative ring in the year 2022.

This paper presents the theoretical results for constructing a new general formula of the first Zagreb index of a zero divisor graph using the set of integers modulo power of primes, $\mathbb{Z}_{p^k}$.

Preliminaries

The concepts and definitions of ring theory, graph theory, and topological index are briefly described in this section.

Definition 1 [22] Ring
A ring R is defined as a nonempty set with two compositions $+,\cdot : R \times R \to R$ with the properties:
(1) $\langle R, +\rangle$ is an abelian group (with the zero element 0);
(2) $\langle R, \cdot\rangle$ is a semigroup;
(3) for all $a, b, c \in R$ the distributive laws are valid:
$$(a+b)c = ac + bc, a(b+c) = ab + ac.$$

Definition 2 [23] Commutative Ring
A ring R is commutative if $ab = ba$ for all $a, b \in R$.

Definition 3 [24] Zero Divisor of a Ring
A nonzero element of a ring is said to be a zero divisor if the product of that nonzero element with another nonzero element of the ring is equal to zero.

Definition 4 [25] Connected Graph
A graph that is in one piece, so that any two vertices are connected by a path.

The representation for the set of all zero divisors in a ring R is denoted by $Z(R)$ while the set of all nonzero zero divisors in R is represented by $Z(R)^*$.

Definition 5 [12] Zero Divisor Graph
Assuming that $Z(R)^*$ represents the set of all zero divisors in a ring R, the zero divisor graph of R, denoted by $\Gamma(R)$, is a graph with vertices representing the zero divisors of R. Two distinct vertices a and b are adjacent if $ab = ba = 0$.

Definition 6 [4] First Zagreb Index
Let Γ be a connected graph. Then, the first Zagreb index of Γ is the sum of the degree of each vertex in the graph Γ to the power of two, written as,

$$M_1(\Gamma) = \sum_{u \in V(\Gamma)} deg(u)^2,$$

where $deg(u)$ represents the number of edges connected to vertex u.

Then handshaking lemma by Gunderson [26] stated that in any graph, the total sum of degrees of all vertices in a graph equals the number of edges in the graph multiplied by two. The purpose of this is to establish the main result in the section that follows.

Lemma 1 [26] Handshaking Lemma
For any graph Γ,

$$\sum_{u \in V(G)} deg(u) = 2|E(G)|.$$

The following result by Juliana [27] will also be needed in the next section.

Proposition 1 [27] Number of Zero Divisors
Let $\mathbb{Z}_n$ be the ring of integers $\mathbb{Z}_n$ where n can be expressed as $n = p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}$, such that $p_1, p_2, \dots p_m$, are distinct prime numbers, and $k_1, k_2, \dots k_m \in \mathbb{N}$. Then, the ring $\mathbb{Z}_n$ has at least a zero divisor, and the number of its zero divisors is

$$n - n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \dots \left(1 - \frac{1}{p_m}\right) - 1.$$

This research focuses on the zero divisor graph of the ring $\mathbb{Z}_n$, namely when $n = p^k$ for prime p and positive integer k.

Results and Discussion

This section presents some findings related to the first Zagreb index of zero divisor graph for the commutative ring $\mathbb{Z}_{p^k}$, denoted as $\Gamma(\mathbb{Z}_{p^k})$. Firstly, we present the set of zero divisors in the commutative ring $\mathbb{Z}_{p^k}$ in the following proposition.

Proposition 2 The set of zero divisors in the commutative ring $\mathbb{Z}_{p^k}$ where prime number p and positive integer k is given by $\{p, 2p, 3p, 4p, \dots, p(p^{k-1} - 1)\}$.

Proof. Let a be a zero divisor of the ring $\mathbb{Z}_{p^k}$ such that $gcd(a, p^k) > 1$, where p is prime and $k \in \mathbb{N}$. Then there exists b in the ring such that $ab = 0$, but $b \neq 0$. Moreover, if a is a zero divisor, then there exists an integer c such that $a = cp^k$, which implies $gcd(a, p^k) > 1$ and taking $b = p^{k-1}$ gives $ab = cp^k(p^{k-1}) = 0 \bmod p^k$. Hence, a zero divisor of the ring can only be a zero divisor if p is divisible by p^k. By the number of its zero divisors in Proposition 1, the set of all such elements is $\{p, 2p, 3p, 4p, \dots, p(p^{k-1} - 1)\}$.

By using Proposition 2, the commutative ring $\mathbb{Z}_{p^k}$ where prime number p and $k = 1$ has no zero divisors. As a result, the commutative ring $\Gamma(\mathbb{Z}_p)$ has no degree and also no vertex. Furthermore, there is one element of zero divisors in the commutative ring $\mathbb{Z}_{p^k}$ where $p = 2$ and $k = 2$. Thus, the commutative ring $\mathbb{Z}_4$ has no degree and only consists of one vertex.

The next step is to determine the degree of each vertex in the zero divisor graph of $\mathbb{Z}_{p^k}$ given in Proposition 3. Then in Proposition 4, the number of vertices is determined. The set operation denoted by $aG = \{a \cdot g \mid g \in G\}$ is used in these two propositions, which are required to prove the main theorem of this paper.

Proposition 3 Let p be a prime number, $k \in \mathbb{N}$ and $a \in \mathbb{Z}_{p^k}$ with $gcd(a, p^k) = p^i$ for $i = 1, 2, \dots, k$. Then, the degree of vertex a in the zero divisor graph of $\mathbb{Z}_{p^k}$ is

$$deg(a) = \begin{cases} p^i - 1, & i \leq \left\lfloor \frac{k-1}{2} \right\rfloor, \\ p^i - 2, & i > \left\lfloor \frac{k-1}{2} \right\rfloor, \end{cases}$$

where $k \geq 2$ for $p = 2$ and $k \geq 2$ for odd primes p, and $\lfloor k \rfloor$ denotes the floor function of k.

Proof. Let $a \in \mathbb{Z}_{p^k}$ with $gcd(a, p^k) = p^i$ and $b \in \mathbb{Z}_{p^k}$ with $gcd(b, p^k) = p^j$ for $i \neq j$. Then a is adjacent to b, that is $ab = 0$ if and only if $j \geq k - i$. So $\left|p^{k-i}\mathbb{Z}_{p^k}\right| = \left|p^{k-i} \cdot \{0,1,2,\dots,p^k - 1\}\right| = p^i - 1$, in which $\{p^{k-i} \cdot 0\} \notin p^{k-i}\mathbb{Z}_{p^k}$. Thus, there are two cases where $k \geq 2$ for $p = 2$ and $k \geq 2$ for odd primes p:

(1) If $i > \left\lfloor \frac{k-1}{2} \right\rfloor$, then $a \in p^{k-i}\mathbb{Z}_{p^k}$. So $deg(a) = p^i - 2$.
(2) If $i \leq \left\lfloor \frac{k-1}{2} \right\rfloor$, then $deg(a) = p^i - 1$.

□

In the following proposition, the number of vertices with the same degree is determined.

Proposition 4 Let $V'_i = \{a \in \Gamma(\mathbb{Z}_{p^k}) : gcd(a, p^k) = p^i\}$, then $|V'_i| = p^{k-i} - p^{k-(i+1)}$ for $1 \leq i \leq k - 1$ where $k \geq 3$ for $p = 2$ and $k \geq 2$ for odd primes p.

Proof. *Let* $a \in V'_i$ for $1 \leq i \leq k - 1$, thus $a \in p^i\mathbb{Z}_{p^k}$ but $a \notin p^{i+1}\mathbb{Z}_{p^k}$. So $|V'_i| = \left|p^i\mathbb{Z}_{p^k}\right| - \left|p^{i+1}\mathbb{Z}_{p^k}\right| = \left|p^i \cdot \{0,1,2,\dots,p^k - 1\}\right| - \left|p^{i+1} \cdot \{0,1,2,\dots,p^k - 1\}\right| = p^{k-i} - p^{k-(i+1)}$ where $k \geq 3$ for $p = 2$ and $k \geq 2$ for odd primes p.

□

Next, the number of edges in the zero divisor graph of the rings $\mathbb{Z}_{p^k}$ is provided. Proposition 5 covers the case in which $k = 1$, and Proposition 6 covers the case in which $p = 2$ and $k = 2$. The rest of the cases are covered in Proposition 7.

Proposition 5 The zero divisor graph of the set of integers modulo p, $\Gamma(\mathbb{Z}_p)$ has zero edges.

Proof. By Definition 3, $\mathbb{Z}_p$ has no zero divisors. Thus, by Definition 5 there are no vertices in the zero divisor graph, $\Gamma(\mathbb{Z}_p)$. Therefore, the number of edges in $\Gamma(\mathbb{Z}_p)$ is zero.

□

Proposition 6 The zero divisor graph of the set of integers modulo 4, $\Gamma(\mathbb{Z}_4)$ has zero edges.

Proof. By Definition 3, $\mathbb{Z}_4$ has only one zero divisor. Thus, by Definition 5 there is one vertex in the zero divisor graph, $\Gamma(\mathbb{Z}_4)$. Therefore, the number of edges in $\Gamma(\mathbb{Z}_4)$ is zero.

□

Lastly, the number of edges in the zero divisor graph for the rings $\mathbb{Z}_{p^k}$ where $k \geq 3$ *for* $p = 2$ and $k \geq 2$ for odd primes p is determined in the following proposition.

Proposition 7 The number of edges of the zero divisor graph for the rings $\mathbb{Z}_{p^k}$, $\Gamma(\mathbb{Z}_{p^k})$ is $\frac{1}{2}\left[(k-1)(p^k - p^{k-1}) - p^{k-1} - p^{\left\lceil \frac{k-1}{2} \right\rceil} + 2\right]$ where $k \geq 3$ *for* $p = 2$ and $k \geq 2$ for odd primes *p.*

Proof. Using Lemma 1, Proposition 3 and Proposition 4, the number of edges of the zero divisor graph for the rings $\mathbb{Z}_{p^k}$,

$$\left|E\left(\Gamma(\mathbb{Z}_{p^k})\right)\right| = \frac{1}{2} \sum_{a \in V\left(\Gamma(\mathbb{Z}_{p^k})\right)} deg(a)$$

$$= \frac{1}{2}\left[\sum_{i=1}^{\left\lfloor \frac{k-1}{2} \right\rfloor} \left(p^{k-i} - p^{k-(i+1)}\right)\left(p^i - 1\right) + \sum_{i=1+\left\lfloor \frac{k-1}{2} \right\rfloor}^{k-1} \left(p^{k-i} - p^{k-(i+1)}\right)\left(p^i - 2\right)\right]$$

$$= \frac{1}{2}\left[\sum_{i=1}^{\left\lfloor\frac{k-1}{2}\right\rfloor}\left(p^{k-i}-p^{k-(i+1)}\right)\left(p^{i}-1\right)+\sum_{i=1+\left\lfloor\frac{k-1}{2}\right\rfloor}^{k-1}\left(p^{k-i}-p^{k-(i+1)}\right)\left(p^{i}-1\right)\right.$$
$$\left.+\sum_{i=1+\left\lfloor\frac{k-1}{2}\right\rfloor}^{k-1}\left(p^{k-i}-p^{k-(i+1)}\right)(-1)\right]$$

$$= \frac{1}{2}\left[\sum_{i=1}^{k-1}\left(p^{k}-p^{k-i}-p^{k-1}+p^{k-i-1}\right)+\sum_{i=1+\left\lfloor\frac{k-1}{2}\right\rfloor}^{k-1}\left(p^{k-i-1}-p^{k-i}\right)\right]$$

$$= \frac{1}{2}\left[(k-1)(p^{k}-p^{k-1})-p^{k-1}-p^{\left\lceil\frac{k-1}{2}\right\rceil}+2\right],$$

where $k \geq 3$ for $p = 2$ and $k \geq 2$ for odd primes.

□

Due to the fact that there are no edges in the zero divisor graph of $\mathbb{Z}_p$ and $\mathbb{Z}_4$ in Proposition 5 and Proposition 6 respectively, this implies that the degree of every vertex is also zero. Therefore, the first Zagreb index of the zero divisor graph for $\mathbb{Z}_p$ and $\mathbb{Z}_4$ is zero.

Next, the following theorem presents our main result.

Theorem 1 The first Zagreb index of the zero divisor graph for $\mathbb{Z}_{p^k}$, $M_1\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right) = 2(p^{k-1}-p^{k})\left(k-1+\left\lceil\frac{k-1}{2}\right\rceil\right)+(p^{k}+1)(p^{k-1}-1)+3\left(p^{\left\lceil\frac{k-1}{2}\right\rceil}-1\right)$ where $k \geq 3$ for $p = 2$ and $k \geq 2$ for odd primes.

Proof. Using Definition 6, Proposition 3 and Proposition 4, the first Zagreb index of the zero divisor graph for $\mathbb{Z}_{p^k}$,

$$M_1\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right) = \sum_{u\in V\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right)}\left(deg(u)\right)^2$$

$$= \sum_{i=1}^{\left\lfloor\frac{k-1}{2}\right\rfloor}\left(p^{k-1}-p^{k-(i+1)}\right)\left(p^{i}-1\right)^2+\sum_{i=1+\left\lfloor\frac{k-1}{2}\right\rfloor}^{k-1}\left(p^{k-i}-p^{k-(i+1)}\right)\left(p^{i}-2\right)^2$$

$$= \sum_{i=1}^{\left\lfloor\frac{k-1}{2}\right\rfloor}\left(p^{k-1}-p^{k-(i+1)}\right)\left(p^{2i}-2p^{i}+1\right)+\sum_{i=1+\left\lfloor\frac{k-1}{2}\right\rfloor}^{k-1}\left(p^{k-i}-p^{k-(i+1)}\right)\left(p^{2i}-2p^{i}+1\right)$$
$$+\sum_{i=1+\left\lfloor\frac{k-1}{2}\right\rfloor}^{k-1}\left(p^{k-i}-p^{k-(i+1)}\right)\left(3-2p^{i}\right)$$

$$= \sum_{i=1}^{k-1}\left(p^{k-i}-p^{k-(i+1)}\right)\left(p^{2i}-2p^{i}+1\right)+\sum_{i=1+\left\lfloor\frac{k-1}{2}\right\rfloor}^{k-1}\left(p^{k-i}-p^{k-(i+1)}\right)(3-2p^{i})$$

$$= 2\sum_{i=1}^{k-1}(p^{k-1}-p^{k})+\;p^{k}\sum_{i=1}^{k-1}\left(p^{i}+\frac{1}{p^{i}}-p^{i-1}-\frac{1}{p^{i+1}}\right)+\;2\sum_{i=1+\left\lfloor\frac{k-1}{2}\right\rfloor}^{k-1}(p^{k-1}-p^{k})$$

$$+\sum_{i=1+\left\lceil\frac{k-1}{2}\right\rceil}^{k-1}\left(\frac{1}{p^i}-\frac{1}{p^{i+1}}\right)$$

$$=2(p^{k-1}-p^k)\left(k-1+\left\lceil\frac{k-1}{2}\right\rceil\right)+(p^k+1)(p^{k-1}-1)+3\left(p^{\left\lceil\frac{k-1}{2}\right\rceil}-1\right),$$

where $k \geq 3$ for $p = 2$ and $k \geq 2$ for odd primes.

□

Two examples of the first Zagreb index for the commutative ring $\mathbb{Z}_{p^k}$ are shown in the following for the case p is even and p is odd, respectively.

Example 1 Let $p = 2$ and $k = 4$, then the zero divisor graph of $\mathbb{Z}_{16}$, $\Gamma(\mathbb{Z}_{16})$, is shown in the following figure.

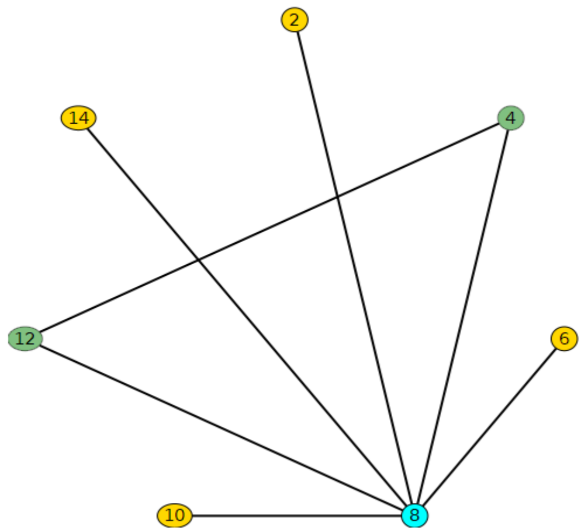

Figure 1. The zero divisor graph for $\mathbb{Z}_{16}$, $\Gamma(\mathbb{Z}_{16})$

By Proposition 1, the number of zero divisors for $\Gamma(\mathbb{Z}_{16})$ can be determined using the formula:

$$p^k - p^k\left(1-\frac{1}{p}\right)-1 = 16 - 16\left(1-\frac{1}{2}\right) - 1 = 7.$$

The vertices of $\Gamma(\mathbb{Z}_{16})$ are the integers set of all zero divisors of $\mathbb{Z}_{16}$, given by the seven elements using Proposition 2 which are $\{2, 4, 6, 8, 10, 12, 14\}$ and the product of two vertices is zero if they are adjacent.

By refering to Figure 1, $\Gamma(\mathbb{Z}_{16})$ has seven edges.

However, by Proposition 7, the number of edges of $\Gamma(\mathbb{Z}_{16})$ can also be calculated as follows:

$$\left|E\left(\Gamma(\mathbb{Z}_{16})\right)\right| = \frac{1}{2}\left[(4-1)(2^4-2^{4-1})-2^{4-1}-2^{\left\lceil\frac{4-1}{2}\right\rceil}+2\right] = 7.$$

By Definition 6, the first Zagreb index can be calculated as follows:

$$\begin{aligned}M_1\left(\Gamma(\mathbb{Z}_{16})\right) &= \sum_{u\in V\left(\Gamma(\mathbb{Z}_{16})\right)} deg(u)^2\\ &= (deg\ (2))^2+(deg\ (4))^2+(deg\ (6))^2+(deg\ (8))^2+(deg\ (10))^2+(deg\ (12))^2\\ &\quad +(deg\ (14))^2\\ &= 48.\end{aligned}$$

Using Theorem 1 is a quick and direct way to determine the value of the first Zagreb index for $\Gamma(\mathbb{Z}_{16})$, which can be shown below:

$$M_1(\Gamma(\mathbb{Z}_{16})) = 2(2^{4-1} - 2^4)\left(4 - 1 + \left\lceil\frac{4-1}{2}\right\rceil\right) + (2^4 + 1)(2^{4-1} - 1) + 3\left(2^{\left\lceil\frac{4-1}{2}\right\rceil} - 1\right) = 48.$$

Example 2 Let $p = 3$ and $k = 3$, then the zero divisor graph of $\mathbb{Z}_{27}$, $\Gamma(\mathbb{Z}_{27})$, is shown in the following figure.

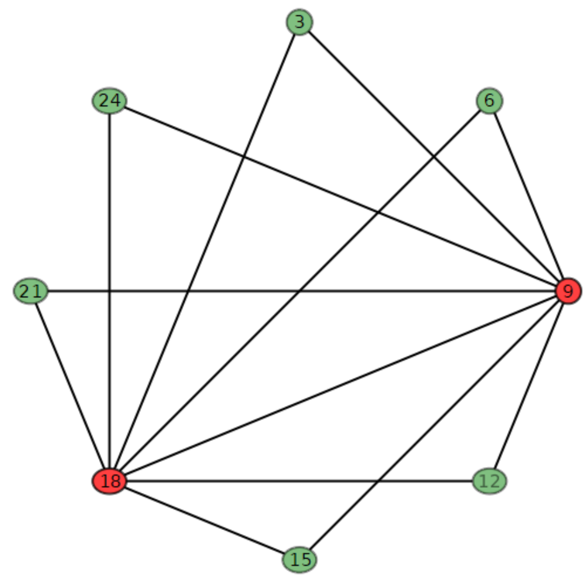

Figure 2. The zero divisor graph for $\mathbb{Z}_{27}$, $\Gamma(\mathbb{Z}_{27})$

By Proposition 1, the number of zero divisors for $\Gamma(\mathbb{Z}_{27})$ can be determined using the formula:

$$p^k - p^k\left(1 - \frac{1}{p}\right) - 1 = 27 - 27\left(1 - \frac{1}{3}\right) - 1 = 8.$$

The vertices of $\Gamma(\mathbb{Z}_{27})$ are the integers set of all zero divisors of $\mathbb{Z}_{27}$, given by the eight elements using Proposition 2 which are $\{3, 6, 9, 12, 15, 18, 21, 24\}$. When two vertices are adjacent in a graph, their multiplication yields zero.

According to Figure 2, $\Gamma(\mathbb{Z}_{27})$ has 13 edges.

However, Proposition 7 can also be used to calculate the number of edges of $\Gamma(\mathbb{Z}_{27})$:

$$\left|E(\Gamma(\mathbb{Z}_{27}))\right| = \frac{1}{2}\left[(3-1)(3^3 - 3^{3-1}) - 3^{3-1} - 3^{\left\lceil\frac{3-1}{2}\right\rceil} + 2\right] = 13.$$

As a result of Definition 6, the calculation of the first Zagreb index consists of the following steps:

$$\begin{aligned} M_1(\Gamma\mathbb{Z}_{27}) &= \sum_{u \in V(\Gamma\mathbb{Z}_{49})} deg(u)^2 \\ &= (deg\ (3))^2 + (deg\ (6))^2 + (deg\ (9))^2 + (deg\ (12))^2 + (deg\ (15))^2 + (deg\ (18))^2 \\ &\quad + (deg\ (21))^2 + (deg\ (24))^2 \\ &= 122. \end{aligned}$$

The first Zagreb index for $\Gamma(\mathbb{Z}_{27})$ can also be determined using Theorem 1 as shown below:

$$M_1(\Gamma(\mathbb{Z}_{27})) = 2(3^{3-1} - 3^3)\left(3 - 1 + \left\lceil\frac{3-1}{2}\right\rceil\right) + (3^3 + 1)(3^{3-1} - 1) + 3\left(3^{\left\lceil\frac{3-1}{2}\right\rceil} - 1\right) = 122.$$

Conclusions

In this paper, the set of zero divisors, degree of each vertex, number of vertices and edges of the zero divisor graph of $\mathbb{Z}_{p^k}$ are determined. Hence, the general formula of the first Zagreb index of the zero divisor graph for the ring of integers modulo power of primes are found. Examples of the zero divisor graph for $\mathbb{Z}_{16}$ and $\mathbb{Z}_{27}$ are provided to illustrate the findings.

Conflicts of Interest

This paper's authors claim no conflict of interest.

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