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M/M/1 Retrial queueing system with negative arrival under non-pre-emptive priority service

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ABSTRACT

Consider a single server retrial queueing system with negative arrival under non-pre-emptive priority service in which three types of customers arrive in a poisson process with arrival rate λ_1 for low priority customers and λ_2 for high priority customers and λ_3 for negative arrival. Low and high priority customers are identified as primary calls. The service times follow an exponential distribution with parameters μ_1 and μ_2 for low and high priority customers. The retrial and negative arrivals are introduced for low priority customers only. Gelenbe (1991) has introduced a new class of queueing processes in which customers are either positive or negative. Positive means a regular customer who is treated in the usual way by a server. Negative customers have the effect of deleting some customer in the queue. In the simplest version, a negative arrival removes an ordinary positive customer or a random batch of positive customers according to some strategy. It is noted that the existence of a flow of negative arrivals provides a control mechanism to control excessive congestion at the retrial group and also assume that the negative customers only act when the server is busy. Let K be the maximum number of waiting spaces for high priority customers in front of the service station. The high priorities customers will be governed by the Non-pre-emptive priority service. The access from the orbit to the service facility is governed by the classical retrial policy. This model is solved by using Matrix geometric Technique. Numerical study have been done for Analysis of Mean number of low priority customers in the orbit (MNCO), Mean number of high priority customers in the queue(MPQL),Truncation level (OCUT),Probability of server free and Probabilities of server busy with low and high priority customers for various values of λ_1 , λ_2 , λ_3 , μ_1 , μ_2 ,σ and k in elaborate manner and also various particular cases of this model have been discussed.

| Queues| Repeated attempts| Negative arrival | Priority service | Matrix Geometric Method |

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1. INTRODUCTION

Queueing systems in which arriving customers who find all servers and waiting positions (if any) occupied may retry for service after a period of time are called **retrial queues** ([3],[4],[7],[16],[18],[19],[25],[26],[30],[31]). Because of the complexity of the retrial queueing models, analytic results are generally difficult to obtain. There are a great number of numerical and approximations methods are available, in this paper solution by Matrix geometric method is emphasised ([9], [10], [14], [20], [24], [29]).

2. DESCRIPTION OF THE QUEUEING SYSTEM

Consider a single server retrial queueing system with negative arrival under non-pre-emptive priority service in which three types of customers arrive in a Poisson process with arrival rate λ_1 for low priority customers and λ_2 for high priority customers and λ_3 for negative arrival ([2], [5], [6], [8], [32], [33]) . Low and high priority customers are identified as **primary calls.** Further assume that the service times follow an exponential distribution with parameters μ_1 and μ_2 for both types of customers. The **retrial and negative arrivals** are introduced for low priority customers only. Let K be the maximum number of waiting spaces for high priority customers in front of the service station.

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2.1 Description of Negative Arrival

Gelenbe (1991) has introduced a new class of queueing processes in which customers are either Positive or Negative. Positive means a regular customer who is treated in the usual way by a server. Negative customers have the effect of deleting some customer in the queue. In the simplest version, a negative arrival removes an ordinary positive customer or a batch of positive customers according to some strategy. It is noted that the existence of a flow of negative arrivals provides a control mechanism to control excessive congestion at the retrial group in tele communication and computer networks. If the primary customer (low priority) finding the server is busy then leaves the service area and re-apply for service after some random time from the orbit. The control mechanism is such that whenever server is busy, an exponential timer is activated. If the timer expires and the server is busy then at random one of the low priority customers which are stored at the retrial pool is automatically removed. A negative arrival has the effect of removing a random customer from the retrial group. The negative customers only act when the server is busy.

If the server is free at the time of a primary call (low/high) arrival, the arriving call begins to be served immediately by the server and customer leaves the system after service completion. Otherwise, if the server is busy then the low priority arriving customer goes to orbit and becomes a source of repeated calls. The pool of sources of repeated calls may be viewed as a sort of queue. Every such source produces a Poisson process of repeated calls with intensity σ. If an incoming repeated call (low) finds the server free, it is served and leaves the system after service, while the source which produced this repeated call disappears. If the server is busy and there are some waiting spaces then the high priority customer can enter into the service station and waits for his service. If there are no waiting spaces then the high priority customers can not enter into the service station and will be lost for the system. Otherwise, the system state does not change.

If the server is engaging with low priority customer and at that time the higher priority customer comes then the high priority customer will get service only after completion of the service of low priority customer who is in service. This type of priority service is called the Non-pre-emptive priority service ([10],[11],[12],[13],[17],[21],[22],[27]). This kind of priority service is followed in this paper.

2.2 Retrial Policy

Most of the queueing system with repeated attempts assume that each customer in the retrial group seeks service independently of each other after a random time exponentially distributed with rate σ so that the probability of repeated attempt during the interval $(t, t + \Delta t)$ given that there were n customers in orbit at time t is $n\sigma\Delta t + O(\Delta t)$. This discipline for access for the server from the retrial group is called **classical retrial rate policy.** The input flow of primary calls (low and high), negative arrivals, interval between repetitions and service times are mutually independent.

3 MATRIX GEOMETRIC METHODS

Let $N(t)$ be the random variable which represents the number of low priority customers in the orbit at time t and $P(t)$ be the random variable which represents the number of high priority customers in the queue (in front of the service station) at time t and $S(t)$ represents the server state at time t. The random process is described as $\{< N(t), P(t), S(t) > / N(t) = 0, 1, 2, \ldots ;\ P(t) = 0, 1, 2, \ldots, k;\ S(t) = 0, 1, 2\}$.

$S(t) = 0$ if server is idle at time t,

$S(t) = 1$ if server busy with low priority customer at time t,

$S(t) = 2$ if server busy with high priority customer at time t.

The possible state spaces are $\{(u, v, w) / u = 0, 1, 2, \ldots ;\ v = 0;\ w = 0, 1, 2\} \cup \{(u, v, w) / u = 0, 1, 2, 3, \ldots ;\ v = 1, 2, 3, \ldots, k;\ w = 1, 2\}$. The infinitesimal generator matrix Q is given below:

$$
Q = \begin{pmatrix}
A_{0,0} & A_0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \dots \\
A_{1,0} & A_{1,1} & A_0 & 0 & 0 & & \dots & 0 & 0 & \dots \\
0 & A_{2,1} & A_{2,2} & A_0 & 0 & 0 & \dots & 0 & 0 & \dots \\
. & . & . & . & . & . & \dots & . & . & \dots \\
. & . & . & . & . & . & \dots & . & . & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & A_{M-1,M-1} & A_0 & \dots \\
0 & 0 & 0 & 0 & 0 & 0 & \dots & A_{M,M-1} & A_{M,M} & \dots \\
. & . & . & . & . & . & \dots & . & . & \dots \\
. & . & . & . & . & . & \dots & . & . & \dots
\end{pmatrix}
$$

The matrices $A_{0,0}$, $A_{n,n\text{-}1}$, $A_{n,n}$, $A_{n,n+1}$ are square matrices of order $2k + 3$.

Notations:

T_1 : $-(\lambda_1 + \lambda_2)$ T_2 : $-(\lambda_1 + \lambda_2 + \lambda_3 + \mu_1)$ T_3 : $-(\lambda_1 + \lambda_2 + \lambda_3 + \mu_2)$ T_4 : $-(\lambda_1 + \lambda_3 + \mu_1)$ T_5 : $-(\lambda_1 + \lambda_3 + \mu_2)$, T_6 : $-(n\sigma + \lambda_1 + \lambda_2)$ T_7 : $-(M\sigma + \lambda_1 + \lambda_2 + \lambda_3)$ T_8 : $-(\lambda_2 + \lambda_3 + \mu_1)$ T_9 : $-(\lambda_2 + \lambda_3 + \mu_2)$ S_1 : $-(\lambda_3 + \mu_1)$, S_2 : $-(\lambda_3 + \mu_2)$ S_3 : $-(\lambda_1 + \lambda_2 + \mu_1)$ S_4 : $-(\lambda_1 + \lambda_2 + \mu_2)$ S_5 : $-(\lambda_1 + \mu_1)$ S_6 : $-(\lambda_1 + \mu_2)$.

$$
A_{0,0} = \begin{pmatrix}
T_1 & \lambda_1 & \lambda_2 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
\mu_1 & S_3 & 0 & \lambda_2 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
\mu_2 & 0 & S_4 & 0 & \lambda_2 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu_1 & S_3 & 0 & \lambda_2 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu_2 & 0 & S_4 & 0 & \lambda_2 & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mu_1 & S_3 & 0 & \dots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & S_4 & \dots & 0 & 0 & 0 & 0 & 0 \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & .. \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & S_3 & 0 & \lambda_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & S_4 & 0 & \lambda_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \mu_1 & S_5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \mu_2 & 0 & S_6
\end{pmatrix}
$$

$A_{n,\,n\text{-}1} = (a_{ij})$ where $a_{11} = 0$, $a_{12} = n\sigma$, $a_{ii} = \lambda_3$, for $i = 2, 3, 4, \dots, 2k + 3$,
$= 0$, otherwise.

$$A_{n,n} = \begin{pmatrix} T_6 & \lambda_1 & \lambda_2 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\ \mu_1 & T_2 & 0 & \lambda_2 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\ \mu_2 & 0 & T_3 & 0 & \lambda_2 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_1 & T_2 & 0 & \lambda_2 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 & T_3 & 0 & \lambda_2 & \ldots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_1 & T_2 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & T_3 & \ldots & 0 & 0 & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & .. \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & T_2 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & T_3 & 0 & \lambda_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & \mu_1 & T_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & \mu_2 & 0 & T_5 \end{pmatrix}$$

$A_{n,n+1} = A_0 = (a_{ij})$ where $a_{11} = 0$, $a_{ii} = \lambda_1$ for $i = 2, 3, 4, \ldots, 2k + 3$,
$= 0$, otherwise.

If the capacity of the orbit is finite say M, then

$$A_{M,M} = \begin{pmatrix} T_7 & \lambda_1 & \lambda_2 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\ \mu_1 & T_8 & 0 & \lambda_2 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\ \mu_2 & 0 & T_9 & 0 & \lambda_2 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_1 & T_8 & 0 & \lambda_2 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 & T_9 & 0 & \lambda_2 & \ldots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu_1 & T_8 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & T_9 & \ldots & 0 & 0 & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & .. \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & T_8 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & T_9 & 0 & \lambda_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & \mu_1 & S_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & \mu_2 & 0 & S_2 \end{pmatrix}$$

Let $\boldsymbol{x}$ be a steady-state probability vector of Q and partitioned as $\boldsymbol{x} = (x(0), x(1), x(2), \ldots)$ and $\boldsymbol{x}$ satisfies

$$\boldsymbol{x}Q = 0\,,\ \boldsymbol{x}e = 1 \qquad (1)$$

where $x(i) = (P_{i00}, P_{i01}, P_{i02}, P_{i11}, P_{i12}, P_{i21}, P_{i22}, \ldots, P_{ik1}, P_{ik2})$ $\quad i = 0,1,2,\ldots$.

4. DIRECT TRUNCATION METHOD

In this method one can truncate the system of equations in (1) for sufficiently large value of the number of customers in the orbit, say M. That is, the orbit size is restricted to M such that any arriving customer finding the orbit full is considered lost. The value of M can be chosen so that the loss probability is small. Due to the intrinsic nature of the system in (1), the only choice available for studying M is through algorithmic methods. While a number of approaches is available for determining the cut-off point , M , The one that seems to perform well (with respect to approximating the system performance measures) is to increase M until the largest individual change in the elements of $\boldsymbol{x}$ for successive values is less than Є a predetermined infinitesimal value.

5. STABILITY CONDITION

Theorem:

The inequality $F\left[\frac{\lambda_1}{\mu_1}-\frac{\lambda_3}{\mu_1}\right] < 1$ where $F=\frac{(1-t^k)}{(1-x)(1-\pi_{2k})+x\pi_{2k+1}}$, $x = \lambda_2/\mu_2$, $y = \mu_1/\mu_2$ and $t = x/(x+y)$ is the necessary and sufficient condition for the system to be stable ([1], [15], [23], [28]). As $k \to\infty$ the above stability condition becomes $\left[\frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2} - \frac{\lambda_3}{\mu_1}\right] < 1.$

Proof:

Let Q be an infinitesimal generator matrix for the queueing system (without retrial).
The stationary probability vector X satisfies

$$XQ = 0 \quad and \quad Xe=1. \tag{2}$$

Let R be the rate matrix satisfying the equation

$$A_0+RA_1+ R^2A_2 =0. \tag{3}$$

The matrices A_0 , A_1 , A_2 are square matrices of order $2k + 2$. The system is stable if $sp(R)<1$.
The matrix R satisfies sp(R) <1 if and only if

$$\Pi A_0e < \Pi A_2e \tag{4}$$

and $\Pi = (\pi_0, \pi_1, \pi_2, \ldots, \pi_{2k}, \pi_{2k+1})$ satisfies

$$\Pi A = 0, \tag{5}$$

$$\Pi e = 1, \tag{6}$$

where

$$A = A_0 + A_1 + A_2 , \tag{7}$$

$A_0 = \lambda_1 I$, where I is an identity matrix,

$$A_1 = \begin{pmatrix}
T_2 & 0 & \lambda_2 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
0 & T_3 & 0 & \lambda_2 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
0 & \mu_1 & T_2 & 0 & \lambda_2 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
0 & \mu_2 & 0 & T_3 & 0 & \lambda_2 & \ldots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mu_1 & T_2 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mu_2 & 0 & T_3 & \ldots & 0 & 0 & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots. & \ldots & \ldots & \ldots. & \ldots. & \ldots. \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & \mu_1 & T_2 & 0 & \lambda_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & \mu_2 & 0 & T_3 & 0 & \lambda_2 \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & \mu_1 & T_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & \mu_2 & 0 & T_5
\end{pmatrix}$$

$A_2 = (a_{ij})$ where $a_{11}= \mu_1+ \lambda_3$, $a_{21}= \mu_2$, $a_{ii}=\lambda_3$ for $i = 2, 3, 4, \ldots, 2k + 2$.

By substituting A_0 , A_1 , A_2 in equations (5) , (6) and (7) ,

$$\pi_1 = x\pi_0,$$
$$\pi_2 = t\pi_0,$$
$$\pi_3 = x(\pi_1 + \pi_2),$$
$$\pi_4 = t^2\pi_0,$$
$$\pi_5 = x(\pi_3 + \pi_4),$$
$$\pi_6 = t^3\pi_0,$$
$$\pi_7 = x(\pi_5 + \pi_6),$$
$$\pi_8 = t^4\pi_0,$$
$$\pi_9 = x(\pi_7 + \pi_8),$$
$$\ldots \quad \ldots \quad \ldots$$
$$\pi_{2k-1} = x(\pi_{2k-3} + \pi_{2k-2}),$$

$\pi_{2k} = (x/y)\, t^{k-1}\, \pi_0,$

$\pi_{2k+1} = x\pi_{2k-1}.$

From (6), $\pi_0 + \pi_1 + \pi_2 + \pi_3 + \pi_4 + \pi_5 + \ . \ . \ . + \pi_{2k-1} + \pi_{2k} + \pi_{2k+1} = 1.$

By substituting the values of π_i in the above equation,

$$\pi_0 = \frac{(1-t)[(1-x)(1-\pi_{2k}) \ + x \ \ \pi_{2k+1}]}{(1-t^k)}.$$

From (4),

$$\frac{\lambda_1}{\mu_1} < \pi_0\left(1+\frac{x}{y}\right) + \frac{\lambda_3}{\mu_1}.$$

By substituting π_0,

$$F\left[\frac{\lambda_1}{\mu_1}-\frac{\lambda_3}{\mu_1}\right] < 1. \qquad (8)$$

The inequality (8) is also a sufficient condition for the retrial queueing system to be stable. Let Q_n be the number of customers in the orbit after departure n^{th} customer from the service station. The embedded Markov chain $\{Q_n,\ n \geq 0\}$ is ergodic if (8) satisfied. It is readily to see that $\{Q_n,\ n \geq 0\}$ is irreducible and aperiodic. It remains to be proved that $\{Q_n,\ n \geq 0\}$ is positive recurrent. The irreducible and aperiodic Markov chain $\{Q_n,\ n \geq 0\}$ is positive recurrent if $|\psi_i| < \infty$ for all i and $\lim_{i \to \infty} \sup\ \psi_i < 0$ where

$$\psi_i = E(\ Q_{n+1} - Q_n / Q_n = i\ ), \qquad (\ i = 0,1,2,3,\ .\ .\ .\ ),$$

$$\psi_i = F\left[\frac{\lambda_1}{\mu_1}-\frac{\lambda_3}{\mu_1}\right] - \frac{i\sigma}{\lambda_1+\lambda_2+i\sigma}\ .$$

If (8) satisfied, then $|\psi_i| < \infty$ for all i and $\lim_{i \to \infty} \sup\ \psi_i < 0$. Therefore the embedded Markov chain $\{Q_n,\ n \geq 0\}$ is erogdic. If $K \to \infty$ then $\pi_{2k} \to 0$ and $\pi_{2k+1} \to 0$ ***and*** $t^k \to 0$. So the above stability condition becomes $\left[\frac{\lambda_1}{\mu_1}+\frac{\lambda_2}{\mu_2}-\frac{\lambda_3}{\mu_1}\right] < 1.$

6. ANALYSIS OF STEADY STATE PROBABILITIES

The Direct Truncation Method is used for finding the steady state probability vector $\boldsymbol{x}$. Let M denote the cut-off point or truncation level. The steady state probability vector $\boldsymbol{x}^{(M)}$ is now partitioned as $\boldsymbol{x}^{(M)} = (x(0), x(1), x(2), \ldots, x(M))$ and $\boldsymbol{x}^{(M)}$ satiesfies $\boldsymbol{x}^{(M)} Q = 0$, $\boldsymbol{x}^{(M)} e = 1$, where $\boldsymbol{x(i)} = (P_{i00}, P_{i01}, P_{i02}, P_{i11}, P_{i12}, P_{i21}, P_{i22}, \ldots, P_{ik1}, P_{ik2})$; $i = 0, 1, 2, 3, \ldots, M$.

The above system of equations is solved exploiting the special structure of the co-efficient matrix. It is solved by Numerical method such as GAUSS-JORDAN elementary transformation method. Since there is no clear cut choice for M, start the iterative process by taking, say $M = 1$ and increase it until the individual elements of $\boldsymbol{x}$ do not change significantly. That is, if M^* denotes the truncation point then $\| x^{M^*}(i) - x^{M^*-1}(i)\|_\infty < \varepsilon$ where ε is an infinitesimal quantity.

7. SPECIAL CASES

1. This model becomes single server retrial queueing with non-pre-emptive priority service if $\lambda_3 \to 0$.
2. This model becomes Single Server Retrial queueing system and coincides with analytic solutions given by Falin and Templeton for various values of λ_1, $(\lambda_2 \to 0)$, $(\lambda_3 \to 0)$, μ_1, $(\mu_2 \to \infty)$, σ and K large.
3. This model becomes Single Server Standard Queueing System and coincides with standard results if $(\lambda_2 \to 0)$, $(\mu_2 \to \infty)$, $(\lambda_3 \to 0)$ and $(\sigma \to \infty)$.
4. This model becomes Single Server Standard queueing system with finite capacity if $(\lambda_1 \to 0)$, $(\mu_1 \to \infty)$,

λ_2, μ_2 *and K.*

8. SYSTEMS PERFORMANCE MEASURES (FOR CLASSICAL RETRIAL POLICY)

The following system measures can be study with the probabilities obtained by using direct truncation method for various values of $\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \sigma$ and K.

a) The probability mass function of server state

Let $S(t)$ be the random variable which represents the server state at time t.

S:	0_{idle}	1_{low}	2_{high}
P:	$\sum_{i=0}^{\infty} p(i,0,0)$	$\sum_{i=0}^{\infty}\sum_{j=0}^{k} p(i,j,1)$	$\sum_{i=0}^{\infty}\sum_{j=0}^{k} p(i,j,2)$

b) The probability mass function of number of customers(low) in the orbit

Let $X(t)$ be the random variable which represents the number of low priority customers in the orbit. In this model the capacity of the orbit is infinite so $X(t)$ takes the values 0, 1, 2, 3, 4, 5,… .

Number of low priority customers (orbit)	Probability
i	$\sum_{j=0}^{k}\sum_{l=1}^{2} p(i,j,l) + p(i,0,0) \quad (i=0,1,2,\ldots).$

c) The Probability mass function of number of high priority customers(queue)

Let $P(t)$ be number of high priority customers in the queue at time t. In this model, the capacity of high priority customers in the queue is finite and $P(t)$ takes the values 0, 1, 2, 3…K.

Number of high priority customers (queue)	Probability
0	$\sum_{i=0}^{\infty}\sum_{l=0}^{2} p(i,0,l)$,
j	$\sum_{i=0}^{\infty}\sum_{l=1}^{2} p(i,j,l) \quad (j=1,2\ldots,k).$

d) The Mean number of high priority customers in the queue

$$\text{MNHP} = \sum_{j=1}^{k} j*\left(\sum_{i=0}^{\infty}\sum_{l=1}^{2} p(i,j,l)\right).$$

e) The Mean number of low priority customers in the orbit

$$\text{MNCO} = \sum_{i=0}^{\infty} i*\left(\sum_{j=0}^{k}\sum_{l=1}^{2} p(i,j,l) + \text{p}(i,0,0)\right)).$$

f) The probability that the orbiting customer (low) is blocked

$$\text{Blocking Probability} = \sum_{i=1}^{\infty}\sum_{j=0}^{k}\sum_{l=1}^{2} p(i,j,l)$$

g) The probability that an arriving customer(low/high) enter into service station immediately

$$\text{PSI} = \sum_{i=0}^{\infty} p(i,0,0)$$

9. NUMERICAL STUDY

MNCO : Mean Number of customers in the orbit,
MPQL : Mean Number of high priority customers in front of the service station,
P0 : Probability that the server is idle,
P1 : Probability that the server is busy with low priority customers,
P2 : Probability that the server is busy with high priority customers,
σ : Retrial rate from the orbit to the service station .

The values of λ_1, λ_2, λ_3, μ_1, μ_2 are subjected to the stability condition discussed in Section 5.

From the following tables,

- Mean number of low priority customers in the orbit decreases as σ increases.
- Probabilities P_0 and P_1 depend on σ.
- As σ increases, P_0 decreases and P_1 increases.
- As K increases, p_2 tends to λ_2/μ_2.

Table 1: Mean number of customers in the orbit and mean queue length of high priority queue for $\lambda_1 = 10$, $\lambda_2 = 5$, $\lambda_3 = 5$, $\mu_1 = 20$, $\mu_2 = 25$, $K = 2$ and various values of σ.

σ	*P0*	*P1*	*P2*	*MNCO*	*MPQL*
10	0.4083	0.3965	0.1952	1.4381	0.1536
20	0.3978	0.4072	0.1951	1.0529	0.1565
30	0.3930	0.4120	0.1950	0.9128	0.1578
40	0.3902	0.4148	0.1950	0.8397	0.1586
50	0.3885	0.4166	0.1950	0.7947	0.1591
60	0.3872	0.4178	0.1950	0.7642	0.1594
70	0.3863	0.4187	0.1950	0.7421	0.1597
80	0.3856	0.4195	0.1950	0.7253	0.1598
90	0.3850	0.4200	0.1949	0.7122	0.1600
100	0.3846	0.4205	0.1949	0.7017	0.1601
200	0.3824	0.4226	0.1949	0.6534	0.1607
300	0.3817	0.4234	0.1949	0.6370	0.1609
400	0.3813	0.4238	0.1949	0.6288	0.1610
500	0.3811	0.4240	0.1949	0.6238	0.1611
600	0.3809	0.4241	0.1949	0.6205	0.1611
700	0.3808	0.4243	0.1949	0.6182	0.1612
800	0.3808	0.4243	0.1949	0.6164	0.1612
900	0.3807	0.4244	0.1949	0.6150	0.1612
1000	0.3806	0.4245	0.1949	0.6139	0.1612
2000	0.3804	0.4247	0.1949	0.6089	0.1613
3000	0.3803	0.4248	0.1949	0.6072	0.1613
4000	0.3803	0.4248	0.1949	0.6064	0.1613
5000	0.3803	0.4248	0.1949	0.6059	0.1613
6000	0.3803	0.4248	0.1949	0.6056	0.1613
7000	0.3802	0.4249	0.1949	0.6053	0.1613
8000	0.3802	0.4249	0.1949	0.6051	0.1613
9000	0.3802	0.4249	0.1949	0.6050	0.1613

Table 2: Mean number of customers in the orbit and mean queue length of high priority queue for $\lambda_1 = 10$, $\lambda_2 = 5$, $\lambda_3 = 5$, $\mu_1 = 20$, $\mu_2 = 25$, $K = 3$ and various values of σ.

Sigma	$P0$	$P1$	$P2$	MNCO	MPQL
10	0.4055	0.3955	0.1989	1.4633	0.1678
20	0.3949	0.4062	0.1989	1.0750	0.1710
30	0.3901	0.4110	0.1989	0.9341	0.1725
40	0.3874	0.4137	0.1989	0.8605	0.1733
50	0.3856	0.4155	0.1989	0.8151	0.1738
60	0.3844	0.4168	0.1989	0.7844	0.1742
70	0.3834	0.4177	0.1989	0.7621	0.1745
80	0.3827	0.4184	0.1989	0.7453	0.1747
90	0.3822	0.4190	0.1989	0.7321	0.1749
100	0.3817	0.4194	0.1989	0.7215	0.1750
200	0.3796	0.4215	0.1989	0.6730	0.1756
300	0.3788	0.4223	0.1989	0.6565	0.1759
400	0.3784	0.4227	0.1989	0.6482	0.1760
500	0.3782	0.4229	0.1989	0.6433	0.1761
600	0.3781	0.4231	0.1989	0.6399	0.1761
700	0.3780	0.4232	0.1989	0.6375	0.1761
800	0.3779	0.4232	0.1989	0.6358	0.1762
900	0.3778	0.4233	0.1989	0.6344	0.1762
1000	0.3778	0.4234	0.1989	0.6332	0.1762
2000	0.3775	0.4236	0.1989	0.6282	0.1763
3000	0.3774	0.4237	0.1989	0.6265	0.1763
4000	0.3774	0.4237	0.1989	0.6257	0.1763
5000	0.3774	0.4237	0.1989	0.6252	0.1763
6000	0.3774	0.4238	0.1989	0.6249	0.1763
7000	0.3774	0.4238	0.1989	0.6246	0.1763
8000	0.3773	0.4238	0.1989	0.6244	0.1763
9000	0.3773	0.4238	0.1989	0.6243	0.1763

Table 3: Mean number of customers in the orbit and mean queue length of high priority queue for $\lambda_1 = 10$, $\lambda_2 = 5$, $\lambda_3 = 5$, $\mu_1 = 20$, $\mu_2 = 25$, $K = 4$ and various values of σ.

Sigma	***P*0**	***P*1**	***P*2**	***MNCO***	***MPQL***
10	0.4049	0.3953	0.1998	1.4697	0.1720
20	0.3943	0.4059	0.1998	1.0809	0.1753
30	0.3895	0.4107	0.1998	0.9396	0.1768
40	0.3867	0.4135	0.1998	0.8659	0.1776
50	0.3850	0.4153	0.1998	0.8206	0.1782
60	0.3837	0.4165	0.1998	0.7898	0.1785
70	0.3828	0.4174	0.1998	0.7675	0.1788
80	0.3821	0.4181	0.1998	0.7506	0.1790
90	0.3815	0.4187	0.1998	0.7374	0.1792
100	0.3811	0.4192	0.1998	0.7268	0.1794
200	0.3789	0.4213	0.1998	0.6782	0.1800
300	0.3782	0.4220	0.1998	0.6617	0.1802
400	0.3778	0.4224	0.1998	0.6535	0.1804
500	0.3776	0.4227	0.1998	0.6485	0.1804
600	0.3774	0.4228	0.1998	0.6451	0.1805
700	0.3773	0.4229	0.1998	0.6427	0.1805
800	0.3772	0.4230	0.1998	0.6409	0.1805
900	0.3772	0.4231	0.1998	0.6396	0.1806
1000	0.3771	0.4231	0.1998	0.6384	0.1806
2000	0.3769	0.4233	0.1998	0.6334	0.1807
3000	0.3768	0.4234	0.1998	0.6317	0.1807
4000	0.3768	0.4235	0.1998	0.6309	0.1807
5000	0.3768	0.4235	0.1998	0.6304	0.1807
6000	0.3767	0.4235	0.1998	0.6300	0.1807
7000	0.3767	0.4235	0.1998	0.6298	0.1807
8000	0.3767	0.4235	0.1998	0.6296	0.1807
9000	0.3767	0.4235	0.1998	0.6295	0.1807

Table 4: Mean number of customers in the orbit and mean queue length of high priority queue for $\lambda_1 = 10$, $\lambda_2 = 5$, $\lambda_3 = 5$, $\mu_1 = 20$, $\mu_2 = 25$, $K = 5$ and various values of σ.

Sigma	***P*0**	***P*1**	***P*2**	***MNCO***	***MPQL***
10	0.4048	0.3953	0.1999	1.4713	0.1731
20	0.3942	0.4059	0.1999	1.0823	0.1764
30	0.3894	0.4107	0.1999	0.9410	0.1779
40	0.3866	0.4134	0.1999	0.8673	0.1788
50	0.3848	0.4152	0.1999	0.8219	0.1793
60	0.3836	0.4165	0.1999	0.7912	0.1797
70	0.3827	0.4174	0.1999	0.7689	0.1800
80	0.3820	0.4181	0.1999	0.7520	0.1802
90	0.3814	0.4187	0.1999	0.7387	0.1804
100	0.3809	0.4191	0.1999	0.7281	0.1806
200	0.3788	0.4212	0.1999	0.6795	0.1812
300	0.3781	0.4220	0.1999	0.6631	0.1814
400	0.3777	0.4224	0.1999	0.6548	0.1816
500	0.3775	0.4226	0.1999	0.6498	0.1816
600	0.3773	0.4227	0.1999	0.6465	0.1817
700	0.3772	0.4229	0.1999	0.6441	0.1817
800	0.3771	0.4229	0.1999	0.6423	0.1817
900	0.3770	0.4230	0.1999	0.6409	0.1818
1000	0.3770	0.4231	0.1999	0.6398	0.1818
2000	0.3768	0.4233	0.1999	0.6347	0.1819
3000	0.3767	0.4234	0.1999	0.6331	0.1819
4000	0.3766	0.4234	0.1999	0.6322	0.1819
5000	0.3766	0.4234	0.1999	0.6317	0.1819
6000	0.3766	0.4234	0.1999	0.6314	0.1819
7000	0.3766	0.4235	0.1999	0.6311	0.1819
8000	0.3766	0.4235	0.1999	0.6310	0.1819
9000	0.3766	0.4235	0.1999	0.6308	0.1819

Table 5: Mean number of customers in the orbit and mean queue length of high queue for $\lambda_1 = 10$, $\lambda_2 = 5$, $\lambda_3 = 5$, $\mu_1 = 20$, $\mu_2 = 25$, $K = 6$ and various values of σ.

Sigma	*P*0	*P*1	*P*2	*MNCO*	*MPQL*
10	0.4047	0.3953	0.2000	1.4717	0.1734
20	0.3941	0.4059	0.2000	1.0827	0.1767
30	0.3893	0.4107	0.2000	0.9414	0.1782
40	0.3866	0.4134	0.2000	0.8677	0.1791
50	0.3848	0.4152	0.2000	0.8223	0.1796
60	0.3836	0.4164	0.2000	0.7915	0.1800
70	0.3826	0.4174	0.2000	0.7692	0.1803
80	0.3819	0.4181	0.2000	0.7524	0.1805
90	0.3814	0.4186	0.2000	0.7391	0.1807
100	0.3809	0.4191	0.2000	0.7285	0.1809
200	0.3788	0.4212	0.2000	0.6799	0.1815
300	0.3780	0.4220	0.2000	0.6634	0.1818
400	0.3777	0.4224	0.2000	0.6551	0.1819
500	0.3774	0.4226	0.2000	0.6501	0.1820
600	0.3773	0.4227	0.2000	0.6468	0.1820
700	0.3772	0.4228	0.2000	0.6444	0.1820
800	0.3771	0.4229	0.2000	0.6426	0.1821
900	0.3770	0.4230	0.2000	0.6412	0.1821
1000	0.3770	0.4230	0.2000	0.6401	0.1821
2000	0.3767	0.4233	0.2000	0.6351	0.1822
3000	0.3767	0.4234	0.2000	0.6334	0.1822
4000	0.3766	0.4234	0.2000	0.6325	0.1822
5000	0.3766	0.4234	0.2000	0.6320	0.1822
6000	0.3766	0.4234	0.2000	0.6317	0.1822
7000	0.3766	0.4234	0.2000	0.6315	0.1822
8000	0.3766	0.4235	0.2000	0.6313	0.1822
9000	0.3766	0.4235	0.2000	0.6311	0.1822

Table 6: Mean number of customers in the orbit and mean queue length of high priority queue for $\lambda_1 = 10$, $\lambda_2 = 5$, $\lambda_3 = 5$, $\mu_1 = 20$, $\mu_2 = 25$, $K = 7$ and various values of σ.

Sigma	*P*0	*P*1	*P*2	*MNCO*	*MPQL*
10	0.4047	0.3953	0.2000	1.4717	0.1735
20	0.3941	0.4059	0.2000	1.0828	0.1768
30	0.3893	0.4107	0.2000	0.9415	0.1783
40	0.3866	0.4134	0.2000	0.8678	0.1792
50	0.3848	0.4152	0.2000	0.8224	0.1797
60	0.3836	0.4164	0.2000	0.7916	0.1801
70	0.3826	0.4174	0.2000	0.7693	0.1804
80	0.3819	0.4181	0.2000	0.7524	0.1806
90	0.3814	0.4186	0.2000	0.7392	0.1808
100	0.3809	0.4191	0.2000	0.7285	0.1809
200	0.3788	0.4212	0.2000	0.6800	0.1816
300	0.3780	0.4220	0.2000	0.6635	0.1818
400	0.3777	0.4224	0.2000	0.6552	0.1820
500	0.3774	0.4226	0.2000	0.6502	0.1820
600	0.3773	0.4227	0.2000	0.6469	0.1821
700	0.3772	0.4228	0.2000	0.6445	0.1821
800	0.3771	0.4229	0.2000	0.6427	0.1821
900	0.3770	0.4230	0.2000	0.6413	0.1822
1000	0.3770	0.4230	0.2000	0.6402	0.1822
2000	0.3767	0.4233	0.2000	0.6351	0.1822
3000	0.3766	0.4234	0.2000	0.6335	0.1823
4000	0.3766	0.4234	0.2000	0.6326	0.1823
5000	0.3766	0.4234	0.2000	0.6321	0.1823
6000	0.3766	0.4234	0.2000	0.6318	0.1823
7000	0.3766	0.4234	0.2000	0.6315	0.1823
8000	0.3766	0.4235	0.2000	0.6314	0.1823
9000	0.3765	0.4235	0.2000	0.6312	0.1823

10. GRAPHICAL STUDY

From the following figures,

1. Mean number of low priority customers in the orbit decreases as σ increases.
2. As σ increases, the mean number of customers in the orbit becomes constant, it shows that the retrial queueing model becomes standard queueing model.

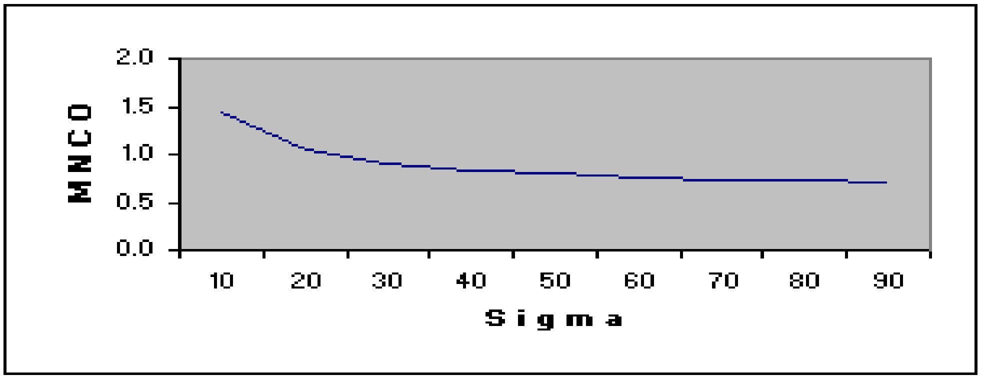

Figure 1: Mean number of low priority customers in the orbit for $\lambda_1 = 10$, $\lambda_2 = 5$, $\lambda_3 = 5$, $\mu_1 = 20$, $\mu_2 = 25$, K=2 and σ varies from 10 to 90.

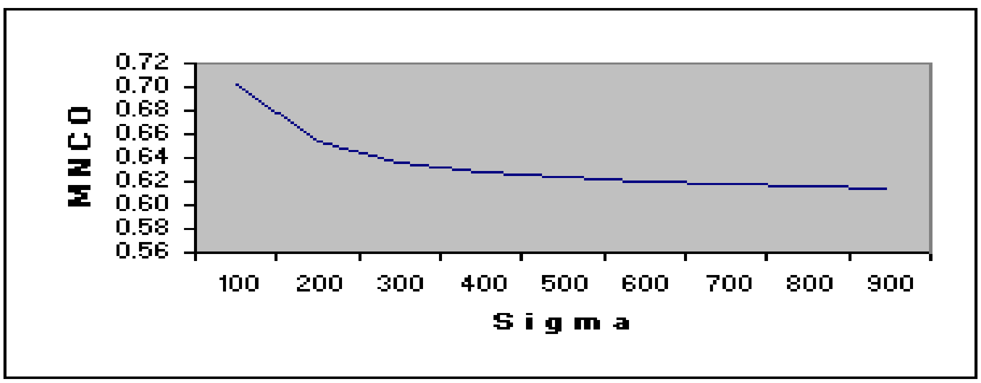

Figure 2: Mean number of low priority customers in the orbit for $\lambda_1 = 10$, $\lambda_2 = 5$, $\lambda_3 = 5$, $\mu_1 = 20$, $\mu_2 = 25$, $K = 2$ and σ varies from 100 to 900.

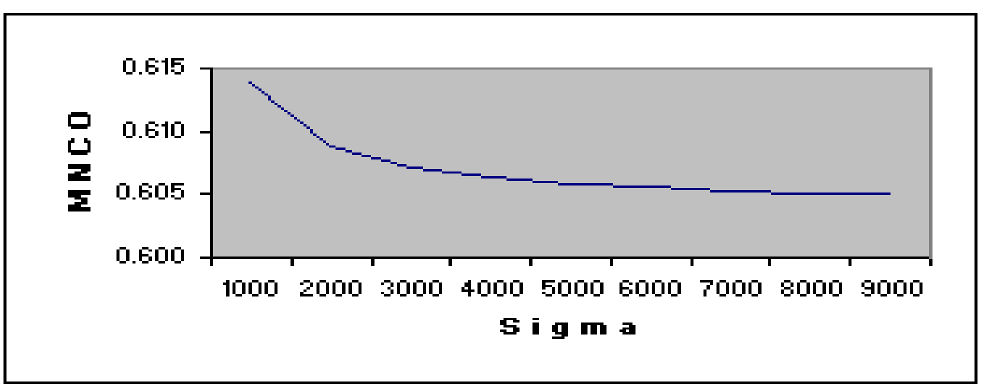

Figure 3: Mean number of low priority customers in the orbit for $\lambda_1 = 10$, $\lambda_2 = 5$, $\lambda_3 = 5$, $\mu_1 = 20$, $\mu_2 = 25$, K=2 and σ varies from 1000 to 9000.

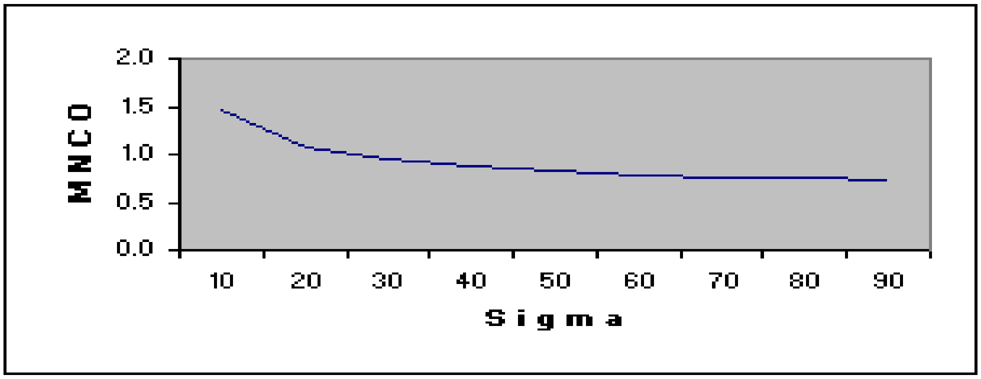

Figure 4: Mean number of low priority customers in the orbit for $\lambda_1 = 10$, $\lambda_2 = 5$, $\lambda_3 = 5$, $\mu_1 = 20$, $\mu_2 = 25$, K=4 and σ varies from 10 to 90.

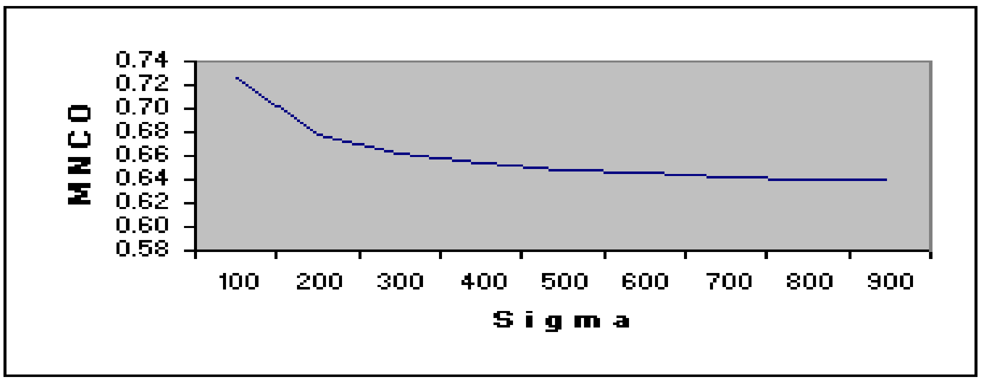

Figure 5: Mean number of low priority customers in the orbit for $\lambda_1 = 10$, $\lambda_2 = 5$, $\lambda_3 = 5$, $\mu_1 = 20$, $\mu_2 = 25$, K=4 and σ varies from 100 to 900.

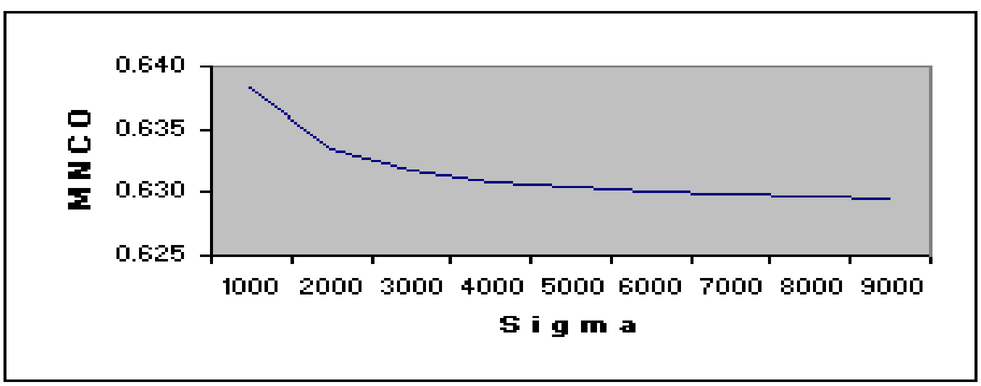

Figure 6: Mean number of low priority customers in the orbit for $\lambda_1 = 10$, $\lambda_2 = 5$, $\lambda_3 = 5$, $\mu_1 = 20$, $\mu_2 = 25$, K=4 and σ varies from 1000 to 9000.

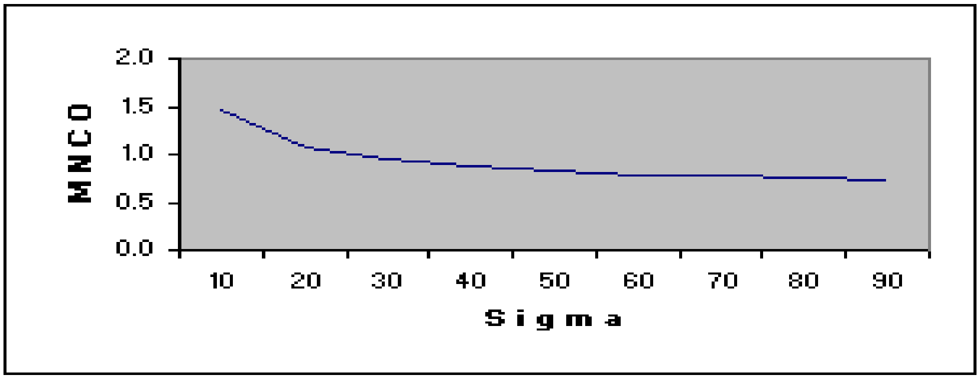

Figure 7: Mean number of low priority customers in the orbit for $\lambda_1 = 10$, $\lambda_2 = 5$, $\lambda_3 = 5$, $\mu_1 = 20$, $\mu_2 = 25$, $K = 6$ and σ varies from 10 to 90.

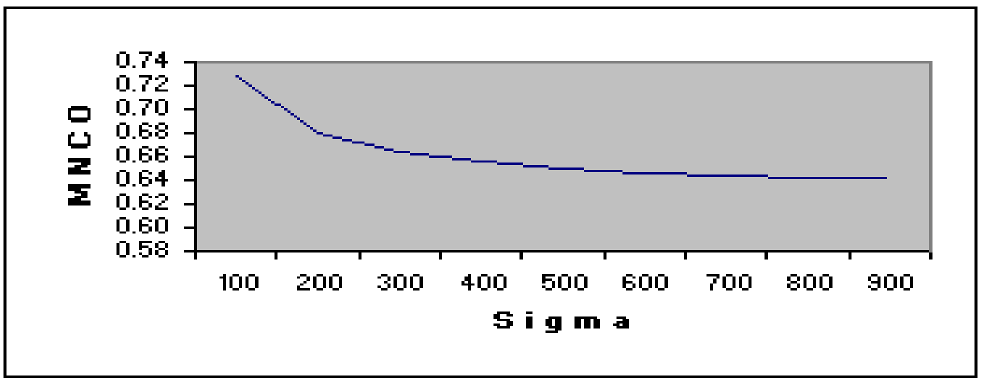

Figure 8: Mean number of low priority customers in the orbit for $\lambda_1 = 10$, $\lambda_2 = 5$, $\lambda_3 = 5$, $\mu_1 = 20$, $\mu_2 = 25$, $K = 6$ and σ varies from 100 to 900.

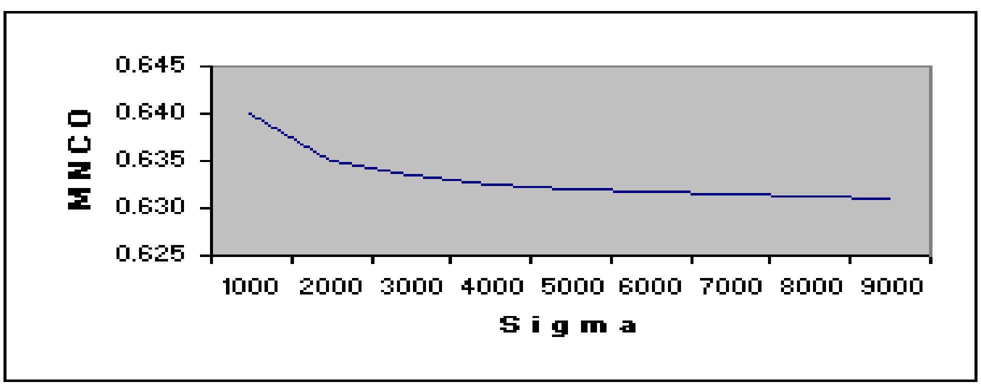

Figure 9: Mean number of low priority customers in the orbit for $\lambda_1 = 10$, $\lambda_2 = 5$, $\lambda_3 = 5$, $\mu_1 = 20$, $\mu_2 = 25$, $K = 6$ and σ varies from 1000 to 9000.

11. CONCLUSIONS

It is observed from Sections 9 and 10 that mean number of low priority customers in the orbit decreases as the retrial rate increases, the probabilities for the server being idle, busy with low priority customers depend on retrial rate. The various special cases which have been discussed in Section 7 are particular cases of this research work. This research work can further be extended by introducing various vacation policies.

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