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RESEARCH ARTICLE

The Quartic Commutativity Degree of Dihedral Groups

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Abstract The combination of group theory and probability theory was used in studying the connection between the two. In recent years, probability theory has been widely used in solving several difficult problems in group theory. The commutativity degree of a group, $P(G)$ is defined as the probability that two random elements in a group commute. In addition, there exist a generalization of commutativity degree of a group which is the n-th power commutativity degree of a group and it is defined as the probability of the n-th power of two random elements in a group commute. Some previous studies have been found for n equal to 2 and 3 and both probabilities are called as squared commutativity degree and cubed commutativity degree respectively. In this research, the n-th power commutativity degree is determined for n equal to 4, called as quartic commutativity degree and some generalization formulas have been obtained. However, this research focuses only on the dihedral groups.

Keywords: Dihedral group, Quartic commutativity degree.

Introduction

A group is called an abelian group if every pair of its elements commute. It means that for a group G, $ab=ba$ for all $a,b\in G$. However, not all groups are abelian, thus are called non-abelian groups. The commutativity degree that is defined on finite groups is a simple tool that measures how much a group is close or far from being abelian. Obviously, the commutativity degree of any group is equal to one if and only if the group is abelian.

The commutativity degree, which is denoted by $P(G)$, is the probability that two elements of a group G, chosen randomly with replacement, commute. As a result of several attempts to develop the concept of the commutativity degree, this concept has been generalized in a number of ways.

The theory of commutativity degree in group theory is one of the oldest areas in group theory and plays a major role in determining the abelianness of the group. It has been attracted by many researchers and it is studied in various directions. Many papers by Miller (1944), Lescot (1995) and Rusin (1979), give explicit formulas of $P(G)$ for some particular finite groups G. The concept of commutativity degree can be generalized and modified in many directions. For instance, two subgroups H and K of G permute if $HK=KH$. Hence, by changing the role of elements to subgroups in a finite group, one can obtain a modification of the commutativity degree of a finite group. If G is a finite group, then the commutativity degree of G, denoted by $P(G)$, is the probability that two randomly chosen elements of G commute. The first appearance of this concept was in 1944 by Miller who distributed a list of open problems related to commutativity degree and its generalizations. Then, the idea to compute $P(G)$ for symmetric groups

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has been introduced by Erdos and Turan (1968) at the end of the 60s.

Mohd Ali and Sarmin (2010) extended the definition of commutativity degree of a group and defined a new generalization of this degree which is called the n-th commutativity degree of a group G, $P_n(G)$ where it is equal to the probability that the n-th power of a random element commutes with another random element from the same group. They also determined $P_n(G)$ for 2 generator 2-groups of nilpotency class two.

A year later, Erfanian *et al*. (2011) gave the relative case of n-th commutativity degree. They identified the probability that the n-th power of a random element of a subgroup, H commutes with another random element of a group G, denoted as $P_n(H,G)$. Meanwhile, Abd Rhani *et al.* (2019) extended the concept of relative commutativity degree by introducing the multiplicative degree of a group G denoted as $P_{mul}(G)$ where it is defined as the probability that the product of a pair of elements chosen randomly from a group G is in the given subgroup of G. They computed the probability for some finite groups. In the same year, they introduced the concept of co-prime probability which is defined as the probability of the order of two random elements in the group are relatively prime. The probability is determined for *p*-groups, where *p* is prime number.

Later in 2020, Muhammad *et al*. released an article of a study on the proof of properties of dihedral group and its commutative elements. The purpose of the research is to prove the properties of elements in the dihedral group using the permutation function and to investigate which elements are commutative in the dihedral group.

Recent studies have been done by Zulkifli *et al*. (2021) on the generalization of the n-th coprime probability and its graph. The *n*-th coprime probability of a group is the extension from the co-prime probability and the coprime graph is a graph whose vertices are members of a group, and two unique vertices are adjacent if and only if the greatest common divisor of the first vertex's order and the second vertex's order is one. More recently, Mohd Ali *et al*. (2022) introduced two new notions called the order product prime probability and the order product prime commutativity degree of groups where they focused on finite dihedral groups and *p*-groups where *p* is prime.

This research is the extension of the n-th power commutativity degree which has been introduced by Abdul Hamid *et al*. (2016) and it is defined as the probability that the *n*-th power of a random pair of elements in the group commute, denoted as $P^n(G)$. If $n=1$, the probability is equivalent to $P(G)$ which is called the commutativity degree. The determination of n-th power commutativity degree has been obtained for dihedral group where both are called as squared commutativity degree and cubed commutativity degree (Abdul Hamid et. al, 2016). Therefore, this research continues the previous studies where $P^n(G)$ is determined for $n=4$ and G is dihedral group. Here $P^4(G)$ is called the quartic commutativity degree.

Materials and Methods

Some basic concepts and important definitions which are the notions of the commutativity degree and its generalization are stated as follows.

Definition 1 The Commutativity Degree of a Group G (Miller, 1944)
Let G be a finite group. The commutativity degree of a group G is denoted by $P(G)$ which is

$$P(G)=\frac{\left|\{(x,y)\in G\times G \mid xy=yx\}\right|}{|G|^2}.$$

Definition 2 The n-th Commutativity Degree of a Group G (Mohd Ali and Sarmin, 2010)
Let G be a finite group. The n-th commutativity degree of a group G is given as

$$P_n(G)=\frac{\left|\left\{(x,y)\in G\times G \mid x^n y = yx^n\right\}\right|}{|G|^2}.$$

Definition 3 Dihedral Groups of Degree n (Gallian, 1986)
For $n\geq 3$, D_n is denoted as the set of symmetries of a regular n-gon. Furthermore, the order of D_n is $2n$ or equivalently $|D_n|=2n$. The Dihedral groups, D_n can be represented in a form of generators and relations given in the following representation:

$$D_n=\left\langle a,b \mid a^n=b^2=1, ba=a^{-1}b\right\rangle.$$

Definition 4 n-th Centralizer of a in G (Mashkouri and Taeri, 2011)
Let a be a fixed element of a group G. The n-th centralizer of a in G, $C_G(a)$ is the set of all elements in G that commute with a^n. In symbols, $C_G^n(a)=\left\{g\in G \middle| ga^n=a^n g\right\}=C_G(a^n)$.

Here $C_G^n(a)$ is a subgroup of G and $\bigcap_{a\in G} C_G^n(a)=C_G(G^n)$, where $G^n=\left\{a^n \middle| a\in G\right\}$. Now define $T_G^n(a)=\left\{g\in G \middle| (ga)^n=(ag)^n\right\}$ and $T_G^n(G)=\bigcap_{a\in G} T_G^n(a)$. It is easy to see that $T_G^n(a)$ may not be a subgroup of G. But it can be seen easily that $T_G^n(G)=C_G(G^n)$ and so $T_G^n(G)$ is a normal subgroup of G. To prove $T_G^n(G)\subseteq C_G(G^n)$, let $a\in T_G^n(G)$. Then for all $g\in G$, $(ag)^n=(ga)^n$. Therefore $\left(a(a^{-1}g)\right)^n=\left((a^{-1}g)a\right)^n$ and so $g^n=a^{-1}g^n a$. Hence $ag^n=g^n a$ and $a\in C_G(G^n)$. To see $C_G(G^n)\subseteq T_G^n(G)$, let $a\in C_G(G^n)$. Then for all $g\in G$, $(ag)^n=(ga)^n$. Therefore $a(ag)^n=(ag)^n a$ and so $(ag)^n=a^{-1}(ag)^n a$. Hence $(ag)^n(ga)^n$ and $a\in T_G^n(G)$.

Definition 5 n-th Center of a Group (Mashkouri and Taeri, 2011)
The n-th center $Z^n(G)$ of a group G is the n-th power of the set of elements in G that commute with every element of G. In symbols,

$$Z^n(G)=\left\{a\in G \mid (ax)^n=(xa)^n \text{ for all } x \text{ in } G\right\}.$$

Definition 6 n-th Power Commutativity Degree (Abdul Hamid *et al*., 2016)
Let G be a finite group. The n-th power commutativity degree of a group G is defined as

$$P^n(G)=\frac{\left|\left\{(x,y)\in G\times G \mid (xy)^n=(yx)^n\right\}\right|}{|G|^2}.$$

Determination of the Quartic Commutativity Degree

The first step is understanding the meaning and concept of the n-th power commutativity degree of a group and its generalizations. Since this study focuses only for the case $n=4$, a new definition of the probability was introduced which is called as quartic commutativity degree. In group theory, there are many finite groups but in this research, we only consider the dihedral group in determining the quartic commutativity degree. This research continues by reviewing and understanding the characteristic of dihedral group and the presentation group is identified in order to construct the modified Cayley Table for some order of dihedral groups.

There are some approaches on finding the commutativity degree of a group. One of the methods is by using modified Cayley Table, which is called 0,1-Table. MacHale (1974) used the 0,1-Table (or Symmetrical Cayley Table) to find the commutativity degree of a group. In this research, we apply the same method in computing the quartic commutativity degree for some dihedral groups and then found general formula. Each theorem has been proved by applying the basic concepts of commutativity degree to verify the results.

The quartic commutativity is determined by using 0,1-Table which consists of only 0's and 1'. The value 1 is placed to each boxes if $(xy)^4 = (yx)^4$. Otherwise, the value 0 will be assigned in each of the boxes if $(xy)^4 \neq (yx)^4$. For an example, the 0,1-Table for dihedral group of order 6 denoted as D_3 given as follow:

Table 1. 0-1-Table for D_3

*	e	a	a^2	b	ab	a^2b
e	1	1	1	1	1	1
a	1	1	1	1	1	1
a^2	1	1	1	1	1	1
b	1	1	1	1	0	0
ab	1	1	1	0	1	0
a^2b	1	1	1	0	0	1

Table 1 shows pairs of elements that commute with each other. There are 30 pairs of element in D_3 that commute. Since the total ordered pairs of D_3 is 36, thus the quartic commutativity degree of dihedral group of order 6 is 30/36. In mathematical symbol, it is written as follows:

$$P^4(D_3) = \frac{30}{36} = \frac{5}{6}.$$

The computation is continued for some order of dihedral groups.

Results and Discussion

Referring to Definition 6, if $n = 4$, then the n-th power commutativity degree is called the quartic commutativity degree and is mathematically defined as follows.

$$\begin{aligned} P^4(G) &= \frac{\left|\left\{(x,y) \in G \times G : (xy)^4 = (yx)^4\right\}\right|}{|G|^2} \\ &= \frac{1}{|G|^2} \sum_{x \in G} \left|\left\{ y \in G : (xy)^4 = (yx)^4 \right\}\right| \\ &= \frac{1}{|G|^2} \sum_{x \in G} \left| T_G^n(x) \right|, \end{aligned} \quad (1)$$

where $T_G^n(x)$ is defined in the following. Note that $x \in G$ **.**

Definition 7 For any group G and for all $a \in G$, $T_G^n(a)$ is defined as all elements g in G such that $(ga)^n = (ag)^n$, and mathematically,

$$T_G^n(a) = \left\{ g \in G \mid (ga)^n = (ag)^n \right\}.$$

In the following, some properties of $T_G^n(a)$ that will be used in the proving of the main theorems are presented.

Proposition 1 Let G be a finite group. Then $T_G^n(G)$ is a normal subgroup of G.

Proof By Definition 4, $C_G^n(a)$ is a subgroup of G and $\bigcap_{a\in G} C_G^n(a) = C_G^n(G)$. Now, define $T_G^n(a) = \left\{ g \in G \mid (ga)^n = (ag)^n \right\}$ and $T_G^n(G) = \bigcap_{a\in G} T_G^n(a)$ may not be a subgroup of G. To prove that $T_G^n(G)$ is a normal subgroup of G, it is enough to prove that $T_G^n(G) = C_G(G^n)$. Hence we need to prove $T_G^n(G) \subseteq C_G^n(G)$ as well as $C_G^n(G) \subseteq T_G^n(G)$. To prove $T_G^n(G) \subseteq C_G^n(G)$, let $a \in T_G^n(G)$. Then for all $g \in G$, $(ag)^n = (ga)^n$. Now, let $g = a^{-1}g$ since g is arbitrary element. Therefore, $\left(a\left(a^{-1}g\right)\right)^n = \left(\left(a^{-1}g\right)a\right)^n$ implies that $g^n = \left(a^{-1}ga\right)^n = a^{-1}gaa^{-1}ga\ldots a^{-1}ga = a^{-1}g^n a$ and so $g^n = a^{-1}g^n a$. Hence, $ag^n = g^n a$ and $a \in C_G^n(G)$. To see $C_G^n(G) \subseteq T_G^n(G)$, let $a \in C_G(G^n)$. Then for all $g \in G$, $ag^n = g^n a$. Again, since g is an arbitrary element, now, let $g = ag$. Therefore, $a(ag)^n = (ag)^n a$ and so $(ag)^n = a^{-1}(ag)^n a$ implies $(ag)^n = a^{-1}(agagag\ldots ag)a = (ga)^n$. Hence, $(ag)^n = (ga)^n$ and $a \in T_G^n(G)$.

□

Proposition 2 Let G be a finite group and 1 an identity element of G. Then $T_G^n(1) = G$.

Proof Let 1 be an identity element of G, then $(g1) = (1g) = g$ for all $g \in G$ since 1 commutes with every element in G. Thus $(g1)^n = (1g)^n$ for all $g \in G$ implies that $T_G^n(1) = G$. □

Proposition 3 Let G be a finite group and $Z^n(G)$ be the n-th center of G. If $a \in Z^n(G)$ then $T_G^n(a) = G$ for all $a \in G$.

Proof Let $Z^n(G)$ be the n-th center of G and $a \in Z^n(G)$. Thus by Definition 5, $(ga)^n = (ag)^n$ for all $g \in G$. Therefore, by Definition 7**,** $T_G^n(a) = G$**.** □

Proposition 4 Let $G = D_m = \left\langle a, b \mid a^m = b^2 = 1, ba = a^{-1}b \right\rangle$ where 1 is an identity element and $\langle a \rangle$ be a cyclic subgroup of D_m. Then $T_G^n(x) = G$ for all $x \in \langle a \rangle$ where n is even.

Proof Suppose $x \in \langle a \rangle = \left\{1, a, a^2, \ldots, a^{m-1}\right\}$. Let $x = a^i$ and $y = a^j$. Thus, $xy = a^i a^j = a^{i+j} = a^{j+i} = a^j a^i = yx$. Since x and y commute, hence $(xy)^n = (yx)^n$. Now, let $y \in G \setminus \langle a \rangle = \left\{b, ab, a^2b, \ldots, a^{m-1}b\right\}$ i.e $y \in \left\{a^j b \mid j = 0, 1, 2, \ldots, m-1\right\}$. If $x = a^i$ and $y = a^j b$, then

$$\begin{aligned} xy &= a^i a^j b \\ &= a^{i+j} b \in G \setminus \langle a \rangle \end{aligned}$$

and

$$\begin{aligned} yx &= a^j b a^i \\ &= a^j a^{-i} b \quad \text{by group presentation of } D_m, \\ &= a^{j-i} b \in G \setminus \langle a \rangle. \end{aligned}$$

Observe that $(xy)^2=(yx)^2$ for all $x\in\langle a\rangle$ and $y\in G\setminus\langle a\rangle$. This implies $(xy)^n=(yx)^n$ for all $x\in\langle a\rangle$ and $y\in G\setminus\langle a\rangle$ where n is even. Obviously, $y^2=1$ for all $y\in G\setminus\langle a\rangle$ and hence $y^n=1$ for all $y\in G\setminus\langle a\rangle$ where n is even. Thus, $(xy)^n=(yx)^n$ for all $y\in G$ and n is even.

□

The following propositions play an important role in the proof of the main results.

Proposition 5 Let G be the dihedral group of order $2m$ where $m\geq 6$ and m is even. Suppose $x\notin Z^4(G)$ where $Z^4(G)=\left\{a\in G\,|\,(ax)^4=(xa)^4 \text{ for all } x\in G\right\}$. If $m/2$ is odd, then $\left|T_G^4(x)\right|=m+2$.

Proof

Let $G=D_m$ where $|D_m|=2m$ and $m/2$ is odd. Let $A=\left\{1,a,a^2,\ldots,a^{m-1}\right\}=\langle a\rangle$ and $B=\left\langle b,ab,a^2b,\ldots,a^{m-1}b\right\rangle$. By Proposition 4, if $x\in\langle a\rangle$ then $T_G^4(x)=G$. We also have

$$T_G^4(b)=\{e,a,a^2,\ldots,a^{m-1},b,a^{\frac{m}{2}}b\}=T_G^4(a^{\frac{m}{2}}b)$$
$$T_G^4(ab)=\{e,a,a^2,\ldots,a^{m-1},ab,a^{\frac{m}{2}+1}b\}=T_G^4(a^{\frac{m}{2}+1}b)$$
$$T_G^4(a^2b)=\{e,a,a^2,\ldots,a^{m-1},a^2b,a^{\frac{m}{2}+2}b\}=T_G^4(a^{\frac{m}{2}+2}b)$$
$$\vdots$$
$$T_G^4(a^{\frac{m}{2}-1}b)=\{e,a,a^2,\ldots,a^{m-1},a^{\frac{m}{2}-1}b,a^{m-1}b\}=T_G^4(a^{m-1}b),$$

Since for $y\in B$ and all $z\in B$, there are only two pairs of elements which satisfy $(yz)^4=(zy)^4=e$. Note that $Z^4(G)=\bigcap_{x\in G}T_G^4(x)$, therefore $Z^4(G)=\{e,a,a^2,\ldots,a^{m-1}\}$ implies $\left|Z^4(G)\right|=m$ Assume that $x\notin Z^4(G)$, therefore we have

$$\begin{aligned}\left|T_G^4(x)\right|&=\left|Z^4(G)\right|+2\\&=m+2.\end{aligned}$$

The proof then follows.

□

Proposition 6 Let G be the dihedral group of order $2m$ where $m\geq 3$ and m is odd. Then $\left|T_G^4(x)\right|=m+1$ in which $x\notin Z^4(G)$.

Proof

Let $G=D_m$ where $|D_m|=2m$ and m is odd. Let $A=\left\{1,a,a^2,\ldots,a^{m-1}\right\}=\langle a\rangle$ and $B=\left\langle b,ab,a^2b,\ldots,a^{m-1}b\right\rangle$. By Proposition 4, if $x\in\langle a\rangle$ then $T_G^4(x)=G$. We also have

$$T_G^4(b)=\{e,a,a^2,\ldots,a^{m-1},b\},T_G^4(ab)=\{e,a,a^2,\ldots,a^{m-1},ab\},$$
$$T_G^4(a^2b)=\{e,a,a^2,\ldots,a^{m-1},a^2b\},\cdots,T_G^2(a^{\frac{m}{2}-1}b)=\{e,a,a^2,\ldots,a^{m-1},a^{\frac{m}{2}-1}b\}.$$

Since for $y\in B$ and all $z\in B$, there is only one pair of elements which satisfy $(yz)^4=(zy)^4=e$. Note that $Z^4(G)=\bigcap_{x\in G}T_G^4(x)$, $Z^4(G)=\{e,a,a^2,\ldots,a^{m-1}\}$ implies $\left|Z^4(G)\right|=m$ Assume that $x\notin Z^4(G)$ therefore we have

$$\left|T_G^4(x)\right|=\left|Z^4(G)\right|+1=m+1,$$

as claimed.

□

Remarks Note that, if $P=\left\{x\in Z^4(G)\right\}$, thus $|P|=\dfrac{|G|}{2}$. Meanwhile, if $|Q|=\left|\left\{x\notin Z^4(G)\right\}\right|$, thus

$$|Q| = \frac{|G|}{2}. \quad (2)$$

Therefore, $|P| + |Q| = \frac{|G|}{2} + \frac{|G|}{2} = |G|$.

The computation of quartic commutativity degree are divided into two cases, namely for m is even and m is odd. The main results of quartic commutativity degree of dihedral groups are stated in the following two theorems. Theorem 1 is for the case m is even where $m/2$ is odd meanwhile Theorem 2 consider for m is odd.

Theorem 1 Let G be dihedral groups of order $2m$ and m is even. If $m/2$ is odd then the quartic commutativity degree is given by:

$$P^4(G) = \frac{3m+2}{4m}.$$

Proof By Equation (1),

$$\begin{aligned} P^4(G) &= \frac{1}{|G|^2} \sum_{x \in G} \left| T_G^4(x) \right| \\ &= \frac{1}{|G|^2} \left[\sum_{x \in Z^4(G)} \left| T_G^4(x) \right| + \sum_{x \notin Z^4(G)} \left| T_G^4(x) \right| \right] \\ &= \frac{1}{|G|^2} \left[\left| Z^4(G) \right| |G| + \sum_{x \notin Z^4(G)} \left| T_G^4(x) \right| \right]. \end{aligned}$$

By Considering Equation (2),

$$P^4(G) = \frac{1}{|G|^2} \left[\left| Z^4(G) \right| |G| + \frac{|G|}{2} \left(\left| T_G^4(x) \right| \right) \right] \quad (3)$$

By Proposition 5,

$$\begin{aligned} P^4(G) &= \frac{1}{|G|^2} \left[\left| Z^4(G) \right| |G| + \frac{|G|}{2} \left(\frac{|G|}{2} + 2 \right) \right] \\ &= \frac{1}{|G|^2} \left[\left| Z^4(G) \right| |G| + \frac{|G|^2}{4} + |G| \right] \\ &= \frac{\left| Z^4(G) \right|}{|G|} + \frac{1}{|G|} + \frac{1}{4} \\ &= \frac{\left| Z^4(G) \right| + 1}{|G|} + \frac{1}{4} \\ &= \frac{m+1}{2m} + \frac{1}{4} \\ &= \frac{3m+2}{4m} \end{aligned}$$

as claimed.

□

Theorem 2 Let G be dihedral groups of order $2m$. If m is odd and $m \geq 3$ then the quartic commutativity degree is given by:

$$P^4(G)=\frac{3m+1}{4m}.$$

Proof

By considering Equation (2) and Proposition 6, then

$$\begin{aligned}P^4(G)&=\frac{1}{|G|^2}\left[\left|Z^4(G)\right||G|+\frac{|G|}{2}\left(\frac{|G|}{2}+1\right)\right]\\&=\frac{1}{|G|^2}\left[\left|Z^4(G)\right||G|+\frac{|G|^2}{4}+\frac{|G|}{2}\right]\\&=\frac{\left|Z^4(G)\right|}{|G|}+\frac{1}{2|G|}+\frac{1}{4}\\&=\frac{\left|Z^4(G)\right|+\frac{1}{2}}{|G|}+\frac{1}{4}\\&=\frac{3m+1}{4m},\end{aligned}$$

as claimed.

□

Conclusions

In conclusion, this study proposed new theorems in group theory. Based on these theorems, the quartic commutativity degree of dihedral groups can be computed and produce faster results than manual calculation. One can substitute the value of m into the formula and obtain the results directly. The results have been found for m is even and m is odd. However, for m is even, the theorem only focuses for the case $m/2$ is odd. The obtained results can be benefited for computing the quartic commutativity degree of dihedral groups of order that related to the both cases.

Conflicts of Interest

The author(s) declare(s) that there is no conflict of interest regarding the publication of this paper.

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