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RESEARCH ARTICLE

Transition Intensities for Critical Illness: A Study on Canadian Health Data

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Abstract Multiple state model is a mathematical model which is characterised by two important elements, transition intensity and transition probability. Critical illness has increased rapidly which is alarmed by the healthcare experts, and becoming an important concern in society. In this paper, by using the Canadian health data, we provide an estimation of transition intensities from the healthy state to the critical illness state with the application of prevalence rate. We provide a discrete calculation of transition intensities with some mathematical formula discussed by some previous studies. Next, we assume that the transition intensities of critical illnesses and death due to other causes are modelled by Gompertz and Makeham mortality models. We also compare and estimate the transition intensities of critical illnesses and dead due to other causes between these two models using a model selection method. We observe the sensitivity of the Gompertz and Makeham models with the different values of extra mortality γ_i. Lastly, we obtain and present the numerical results of the transition intensities of healthy lives to critical illness with the Canadian health data.

Keywords: Multiple state models, Transition intensities, Gompertz-Makeham, Prevalence rates, Critical illness insurance.

Introduction

A multiple state model is a mathematical model that describes the movement of a transient between states. In multiple state model, transition intensity and transition probability are the key elements on developing the model. Both elements were used to calculate the insurance benefits and premiums. According to the past researches, researchers have proposed methods to estimate the transition intensity conveniently, for example, Christiansen [6], Dickson *et al*. [8], Haberman and Pitacco [15], and Jones [16]. A well-known method used in developing the transition intensity is called as Maximum Likelihood Estimation (MLE) function and it was first introduced by Waters [29]. This method has been applied in the recent actuarial context, for example, Baione and Levantesi [1], Li [17], and Pasaribu *et al*. [22] where the transition intensities were estimated discretely for each age group from a health data. When the collected data are discrete in terms of individual age or grouped age, a continuous mortality model is allowed to be used as a fitting model with the collected data. Many types of mortality model, for example, Gompertz-Makeham, Weibull and Exponential mortality models had been summarised by Forfar [10] where these mortality models are also applicable in the mortality and age population study. As a focus of this paper, we consider Gompertz-Makeham mortality model as a fitting model of the discrete transition intensity.

Gompertz-Makeham mortality model is popularly used to estimate the transition intensity in most of the actuarial practice. Baione and Levantesi [1] estimated the transition intensity of critical illness, dead due to other causes, and dead due to critical illness by using Gompertz mortality model with the applications of prevalence rate to the critical illness insurance. Next, Baione and Levantesi [2] compared the transition intensities of dead due to other causes, and dead due to critical illness with Gompertz and Weibull mortality models by assessing the statistical results of the model such as R-square, adjusted R-square, mean square error (MSE) and root mean square (RMSE). They concluded that the Gompertz mortality model as a better fitting model due to its lowest residual standard error of model. On the other hand, Li [17] followed the research work contributed by Baione and Levantesi [2] and concluded that Gompertz

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mortality model as a better mortality model due to its better goodness of fit using the Canadian health data.

This paper follows the motivation of Baione and Levantesi [1], and Li [17] where it is organised as follows. First, we define the mathematical assumptions used in multiple state modelling for the critical illness insurance. Then, we describe the framework of transition intensities with Gompertz and Makeham mortality models while providing the derivation of the transition probabilities. By using the results obtained, we then perform an empirical analysis of the application to Canadian health data provide by Canadian Chronic Disease Surveillance System (*CCDSS)* [5]. Similar data has been used in Li [17] to investigate the transition intensities with Gompertz and Weibull mortality models.

Review of Previous Study

Actuarial researchers are interested to estimate the transition intensities with some estimation methods. The common approach of estimating transition intensities is called as Maximum likelihood estimation (MLE) method which was introduced by Waters [29]. This method can be applied if there is a sufficient data. For a further extension of MLE method, Baione and Levantesi [1], Forfar *et al*. [9], Li [17] and Pasaribu *et al*. [22] applied MLE method on a continuous mortality model which is act as a graduation (or a fitting) model of transition intensities in their research.

Gompertz, Makeham, generalised Gompertz-Makeham, Lee-Carter and many other mortality models can be acted as a graduation model which has been summarised by Forfar [10]. Gompertz mortality model is a pioneer model introduced by Gompertz [13] which is used to describe the mortality rate in a mortality data. This mortality model captures the mortality rates that depends on age. Gompertz mortality model is also applicable in such biological study, for example, Shklovskii [25] assumed and applied Gompertz mortality model to neutralize the defective cells based on random encounters of hazard substances in the biological mechanism. Nevertheless, Michael [20] also estimated the parameters of Gompertz mortality model by addressing some analytical techniques for the insured population in England and Wales. However, Gompertz mortality model captures the mortality rates that is highly depending on age. Therefore, Makeham [18] added an age-independent parameter into the Gompertz mortality model to further extend the work contributed by Gompertz [13]. This extension of mortality model contributed by Makeham [18] is called as Makeham mortality model which provides a better capture of population study in the mortality data. Several researchers also used Makeham mortality model into their research. For example, Ping and Xiang [25] considered the application of Makeham distribution to the discrete life insurance with some variable interest rates. Salhi *et al*. [24] also considered Makeham mortality model as an adaptive smoothing function to adjust the graduated mortality table by using credibility approach. A generalisation of the combined models of Gompertz and Makeham mortality models is called as Gompertz-Makeham (GM) mortality model. GM mortality model also have discussed thoroughly by Golubev [11] and Pitacco and Tabakova [23]. Golubev [12] also used Strehler-Mildvan correlation method to examine the correlation between the parameters of GM mortality model. Also, Wrycza [35] presented and shown that the parameter of the Gompertz-Makeham mortality model can be associated with a simple expression of life table entropy in demographic study. Other than these three types of mortality model, Lee-Carter mortality model is also popularly used to model and forecast the age population study. This mortality model is introduced by Carter and Lee [4] and it has been demonstrated with some reliability study of the model (see, for example, Girosi and King [14]). However, the Lee-Carter mortality model may not be suitable for a certain data sets due to its nature of the data. Therefore, some improvement of this mortality model in terms of its parameter estimation, robustness, and interpretability have been suggested by De Jong and Tickle [9], De Jong *et al*. [10], Lee *et al*. [22], Park *et al*. [27] and Santolino [33].

From the defined continuous mortality model, it is important for actuarial science researchers to select the most preferred mortality models as a final model in our study. Therefore, a comparison between the mortality models should be made with some statistical method. A statistical method that is usually used to select a final model is known as model selection. Some literature reviews for model selection method are also used by Baione and Levantesi [2], Castellares *et al*. [4], Li [17] and Melnikov and Romaniuk [19].

Definitions and Assumptions

In this section, we provide the mathematical assumptions which will be applied in our transition intensity modelling. To start with the assumption, we first provide the definition of the transition probability and the

transition intensity. Here, let $S(x)$ be the state of a person at age *x* and for any positive value from time 0 to time *T*, we denote as $[0,T]$.

Transition probability defines the probability of a person currently aged *x* who transits from State *i* to State *j* within *t* years, it can be expressed as:

$$ {}_tp_x^{ij} = P\left[S(x+t) = j \mid S(x) = i\right], \text{ for } t \in [0,T],\ i,j \in S,\ i \neq j. $$

Specifically, if a person currently aged *x* and is in State *i* will remain in the same State *i* for the next *t* years, then the transition probability is defined by:

$$ {}_tp_x^{\bar{i}\bar{i}} = P\left[S(x+z) = i \text{ for all } z \in [0,T],\ S(x) = 1\right] \tag{1} $$

Besides, transition intensity is fundament for modelling lifetimes based on its rate of occurrence of events. In a multiple state model, transition intensity, denoted by μ_x^{ij}, is defined as the instantaneous rate of occurrence of a person who currently aged *x* will make a transition from State *i* to State *j*. It can be expressed as:

$$ \mu_x^{ij} = \lim_{t \to 0} \frac{{}_tp_x^{ij}}{t}, \text{ for } t \in [0,T],\ i,j \in S,\ i \neq j. $$

According to the approach mentioned by Waters [29], the transition intensity is also defined as a ratio of the number of observed transitions from State *i* to State *j* over the total observed times spent in State *i*. Therefore, the transition intensity is now expressed as:

$$ \mu_x^{ij} = \frac{d_x^{ij}}{l_x^{(i)}}, \text{ for } i,j \in S,\ i < j,\ i \neq j \tag{2} $$

where d_x^{ij} is the number of people who make a transition from State *i* to State *j* between age *x* and age *x* + *t* and $l_x^{(i)}$ is the number of people who currently aged *x* and is in State *i*.

With the defined transition intensities, the transition probabilities can be solved using the Kolmogorov forward differential equations if the transition intensities are available in the model (see, for example, Dickson el al. [8], Haberman and Pitacco [15]). Therefore, the general formula for the transition probabilities for $i,j \in S$ where $i < j$ are:

$$ {}_tp_x^{\bar{i}\bar{i}} = \exp\left(-\int_0^t \sum_{i \neq j} \mu_{x+u}^{ij}\, du \right) \tag{3} $$

$$ {}_tp_x^{ij} = \int_0^t {}_tp_x^{\bar{i}\bar{i}} \mu_{x+u}^{ij}\, du \tag{4} $$

$$ {}_tp_x^{0i} = \int_0^t {}_up_x^{\overline{00}} \cdot \mu_{x+u}^{0i} \cdot {}_{t-u}p_{x+u}^{\bar{i}\bar{i}}\, du \tag{5} $$

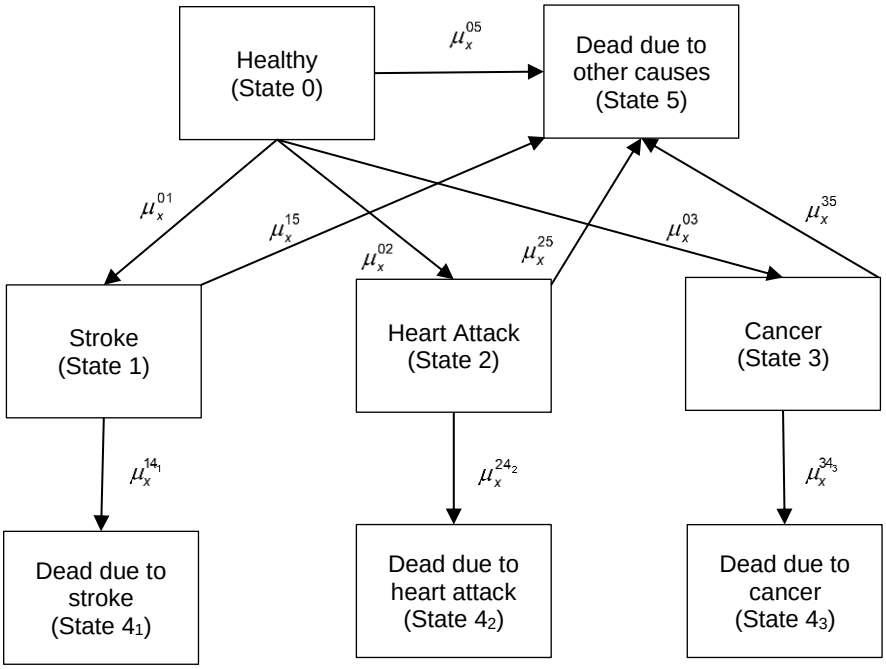

Figure 1. Multiple states and transitions for critical illness insurance model

In this paper, we consider a multiple state model with three types of critical illness, stroke, heart attack and cancer as shown in Figure 1 where State 0 denotes as 'Healthy', State 1 as 'Stroke', State 2 as 'Heart attack', State 3 as 'Cancer', State 4_1 as 'Dead due to stroke', State 4_2 as 'Dead due to heart attack', State 4_3 as 'Dead due to cancer' and State 5 as 'Dead due to other causes' respectively.

We also made the following assumptions of the model:

- Dead due to other causes (State 5) includes all types of death other than stroke (State 1), heart attack (State 2) and cancer (State 3).
- A healthy insured will not make a direct transition to State 4_1, 4_2, 4_3.
- The probability of transitioning from one critical illness to another is zero.
- We do not consider the recovery of the insured in this insurance plan so probability of transitioning from a critical illness to healthy is assumed to be zero.
- Death states are absorbing states where the probabilities of transitioning from these death states to other states are zero.
- It is possible that an insured with cancer dies due to heart attack or stroke, or an insured who survives from heart attack (or stroke) dies due to cancer or stroke (or heart attack). Since it is hard to find the corresponding reliable mortality data, we will ignore these transitions in this research for simplicity.

According to the model in Figure 1 and the equations (3) and (4), for $i = 1,2,3$, the transition probabilities are as follows:

$$ {}_tp_x^{\overline{00}} = \exp\left(-\int_0^t \mu_{x+u}^{01} + \mu_{x+u}^{02} + \mu_{x+u}^{03} + \mu_{x+u}^{05}\, du\right) \tag{6} $$

$$ {}_tp_x^{\overline{ii}} = \exp\left(-\int_0^t \mu_{x+u}^{i4_i} + \mu_{x+u}^{i5}\, du\right) \tag{7} $$

$$ {}_tp_x^{i4_i} = \int_0^t {}_up_x^{\overline{ii}} \mu_{x+u}^{i4_i}\, du \tag{8} $$

Transition Intensities Framework

A direct method on estimating the transition intensities from a critical illness (State *i*) to death due to other causes (State 5) are applicable if there is a complete data available. It is appropriate to introduce extra mortality on μ^{05} to estimate μ^{i5} for critical illness *i*. The extra mortality, denoted by γ_i, is defined as the ratio of the differences between the actual number of deaths who are exposed to risk and the expected number of deaths due to illnesses *i* to the expected number of deaths during a given time period (see Haberman and Pitacco [15]). For example, Dash and Grimshaw [7] assume that the extra mortality as the mortality rate of the critical illness from other causes exceeding that of healthy people.

If the data obtained are incomplete, discrete and aggregated age-group for both mortality and morbidity data, it is appropriate to fit the discrete transition intensities with the continuous mortality models. Suppose the issuing age of the insureds in the critical illness insurance is at age *x*. First, we assume that the transition intensities $\left\{\mu_x^{14_1}, \mu_x^{24_2}, \mu_x^{34_3}\right\}$, μ_x^{05} and $\left\{\mu_x^{15}, \mu_x^{25}, \mu_x^{35}\right\}$ are described by two independent Gompertz-Makeham (*GM*) model.

We define the general formula of the Gompertz-Makeham (*GM*) model of order (r,s) as:

$$GM(r,s) = \sum_{h=1}^{r} \alpha_h x^{h-1} + \exp\left(\sum_{k=1}^{s} \beta_k x^{k-1}\right) \tag{9}$$

where *r* is the polynomial order and *s* is the polynomial order of the exponential, while α and β are the vectors of non-negative parameters.

We would want to obtain the two independent *GM* mortality model. We first fix the parameters r and s of *GM* by considering $r \leq 1$ and $s = 2$. By fixing *r* and *s*, the *GM* model with $(r,s) = (0,2)$ is the Gompertz mortality model and $(r,s) = (1,2)$ is the Makeham mortality model. The solution for $r = 0$ and $s = 2$ may not suitable for all mortality experiences data but to avoid over-parameterisation when the data is scarce (see for example, Brink [3]). This assumption provides a consistent empirical evidence for various data sets on health data. *GM* models generally act as a graduation function on mortality and morbidity data however the choice of concerning must be made carefully as it will make a big difference on modelling results based on the obtained data.

For the transition intensity, denoted by $\mu_x^{i4_i}$, a direct estimation of the direct transition intensity can be obtained by using equation (2). After obtaining the discrete transition intensity, we denote $GM^{ij}(r,s)$ to be the $GM(r,s)$ models from State *i* to State *j*. Now, we assume the mortality model of the transition intensity, denoted by $\mu_x^{i4_i}$ will be fitting with the two independent *GM* models.

Suppose that $\mu_x^{i4_i} \approx GM^{i4_i}(0,2)$. In the formula, let

$$\mu_x^{i4_i} = \exp\left(\beta_1^{i4_i} + \beta_2^{i4_i} x\right) \text{ with } \beta_1^{i4_i} < 0 \text{ and } \beta_2^{i4_i} > 0, \tag{10}$$

where $\beta_1^{i4_i}$ and $\beta_2^{i4_i}$ are the constant parameters for people in State 0.

Also, by $GM^{i4_i}(1,2)$,

$$\mu_x^{i4_i} = \alpha^{i4_i} + \exp\left(\beta_3^{i4_i} + \beta_4^{i4_i} x\right) \text{ with } \beta_3^{i4_i} < 0 \text{ and } \alpha^{i4_i}, \beta_4^{i4_i} > 0, \tag{11}$$

where α^{i4_i}, $\beta_3^{i4_i}$ and $\beta_4^{i4_i}$ are the constant parameters for people in State 0.

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However, the transition intensity, denoted by μ_x^{05} cannot be estimated directly using equation (2) since it consists of all causes of death except stroke, heart attack and cancer. Therefore, by Li [17], it can be expressed as:

$$\mu_x^{05} = \frac{d_x - \sum_{i=1}^{3} d_x^{i4_i}}{l_x + \sum_{i=1}^{3} \gamma_i l_x^{(i)}}, \tag{12}$$

where d_x is the total number of death currently aged x for all causes, $d_x^{i4_i}$ is the number of death currently aged x transits from State i to State 4_i, l_x is the total number of lives currently aged x and $l_x^{(i)}$ is the number of lives currently aged x and is in State i.

By obtaining the discrete transition intensity μ_x^{05} using equation (12), it is assumed to follow a $GM^{05}(0,2)$ model and $GM^{05}(1,2)$ model for all $i = 1,2,3$ where α^{05}, β_1^{05}, β_2^{05}, β_3^{05} and β_4^{05} are the *GM* parameters for critical illness sufferers:

$$\mu_x^{05} = \exp\left(\beta_1^{05} + \beta_2^{05} x\right) \text{ with } \beta_1^{05} < 0 \text{ and } \beta_2^{05} > 0, \tag{13}$$

and

$$\mu_x^{05} = \alpha^{05} + \exp\left(\beta_3^{05} + \beta_4^{05} x\right) \text{ with } \beta_3^{05} < 0 \text{ and } \beta_4^{05} > 0, \tag{14}$$

Meanwhile, for the transition intensity denoted by μ_x^{i5}, we apply the approach proposed by Dash and Grimshaw [7] based on the assumption that the transition intensities of being critically ill people from causes other than critical illness exceeds the transition intensities of healthy ones by an extra mortality of γ_i where $\gamma_i \geq 0$, which is:

$$\mu_x^{i5} = (1+\gamma_i)\mu_x^{05} \text{ for all } i = 1,2,3, \tag{15}$$

For a continuous estimation on the transition intensities μ_x^{i5}, the Gompertz and Makeham mortality models can be expressed as follows:

$$\mu_x^{i5} = \exp\left(\beta_1^{i5} + \beta_2^{i5} x\right) \text{ with } \beta_1^{i5} < 0 \text{ and } \beta_2^{i5} > 0, \tag{16}$$

and

$$\mu_x^{i5} = \alpha^{i5} + \exp\left(\beta_3^{i5} + \beta_4^{i5} x\right) \text{ with } \beta_3^{i5} < 0 \text{ and } \beta_4^{i5} > 0, \tag{17}$$

where the continuous parameters of the models are $\alpha^{i5} \cong (1+\gamma_i)\alpha^{05}$, $\beta_j^{05} \cong \ln(1+\gamma_i)\beta_j^{05}$ for $k = 1,3$ and $\beta_\ell^{i5} \cong \beta_\ell^{05}$ for $\ell = 2,4$.

Before we discuss the methodology used in the transition intensity μ_x^{0i}, we introduce the prevalence rate as follows. We denote the state of a person aged x as $S(x)$. Suppose all insureds currently aged x_0 and is in Healthy state (State 0) where $S(x_0) = 1$. According to Haberman [14], the prevalence rate aged x, denoted by f_x, is defined as the ratio of the number of person who currently aged x and are sick at a specific point in time over the total cohort size at that time. Hence, the prevalence rates of critical illness $f_x^{(i)}$ can be considered as the probability of the insureds currently aged x_0 being critically ill at age x such that $x = x_0 + s$ for $s > 0$ is

$$f_x^{(i)} = P\left[S(x) = i \mid (S(x) = 0 \vee S(x) = i) \wedge S(x_0) = 0\right], \; i = 1,2,3, \tag{18}$$

Consequently, the following equation holds according to Haberman [14]. For $i = 1,2,3$ and $s > 0$,

$$f_{x_0+s}^{(i)} = \frac{{}_sp_{x_0}^{0i}}{{}_sp_{x_0}^{\overline{00}} + {}_sp_{x_0}^{01} + {}_sp_{x_0}^{02} + {}_sp_{x_0}^{03}}, \tag{19}$$

From equation (19), we can expand the prevalence rate for age $x_0 + s + t$, thus we have

$$f_{x_0+s+t}^{(i)} = \frac{{}_{s+t}p_{x_0}^{0i}}{{}_{s+t}p_{x_0}^{\overline{00}} + {}_{s+t}p_{x_0}^{01} + {}_{s+t}p_{x_0}^{02} + {}_{s+t}p_{x_0}^{03}}., \tag{20}$$

where ${}_{s+t}p_{x_0}^{\overline{00}} = {}_sp_{x_0}^{\overline{00}} \cdot {}_tp_{x_0+s}^{\overline{00}}$ and ${}_{s+t}p_{x_0}^{0i} = {}_sp_{x_0}^{\overline{00}}\,{}_tp_{x_0+s}^{0i} + {}_sp_{x_0}^{0i}\,{}_tp_{x_0+s}^{\bar{i}\bar{i}}$.

After we have understand the prevalence rate and defined all the mortality functions, now we are able to determine the transition intensities, denoted by μ_x^{0i}. The transition intensity can be described by a piecewise constant function (for $i = 1,2,3$ and $k = 0,1,2,\ldots,n-1$) as follows (Olivieri [21]):

$$\mu_x^{0i} = \begin{cases} 0 & , x < x_0 \\ \sigma_k^{0i} & , x_k \le x < x_{k+1} \\ \sigma_n^{0i} & , x \ge x_n \end{cases}, \tag{21}$$

where *n* is the number of prevalence rates available from statistical data.

The use of piecewise constant transition intensity has been proposed by Jones [16]. Practically, the transition intensity tends to vary as the age-group interval increases.

A summary table of all the defined mortality functions of the transition intensity μ_x^{ij} for $i < j$ and $i, j \in S$ are presented in Table 1.

Table 1. Summary table of mortality models for transition intensities estimation

Mortality model		Transition intensity	Parameter
$\mu_x^{01}, \mu_x^{02}, \mu_x^{03}$	Constant piecewise function	$\mu_x^{0i} = \begin{cases} 0 & , x < x_0 \\ \sigma_k^{0i} & , x_k \le x < x_{k+1} \\ \sigma_n^{0i} & , x \ge x_n \end{cases}$	σ_k^{0i}
μ_x^{05}	$GM^{05}(0,2)$	$\mu_x^{05} = \exp\left(\beta_1^{05} + \beta_2^{05}x\right)$	β_1^{05} and β_2^{05}
	$GM^{05}(1,2)$	$\mu_x^{05} = \alpha^{05} + \exp\left(\beta_3^{05} + \beta_4^{05}x\right)$	$\alpha^{05}, \beta_3^{05}$ and β_4^{05}
$\mu_x^{14_1}$	$GM^{14_1}(0,2)$	$\mu_x^{14_1} = \exp\left(\beta_1^{14_1} + \beta_2^{14_1}x\right)$	$\beta_1^{14_1}$ and $\beta_2^{14_1}$
	$GM^{14_1}(1,2)$	$\mu_x^{14_1} = \alpha^{14_1} + \exp\left(\beta_3^{14_1} + \beta_4^{14_1}x\right)$	$\alpha^{14_1}, \beta_3^{14_1}$ and $\beta_4^{14_1}$
$\mu_x^{24_2}$	$GM^{24_2}(0,2)$	$\mu_x^{24_2} = \exp\left(\beta_1^{24_2} + \beta_2^{24_2}x\right)$	$\beta_1^{24_2}$ and $\beta_2^{24_2}$
	$GM^{24_2}(1,2)$	$\mu_x^{24_2} = \alpha^{24_2} + \exp\left(\beta_3^{24_2} + \beta_4^{24_2}x\right)$	$\alpha^{24_2}, \beta_3^{24_2}$ and $\beta_4^{24_2}$
$\mu_x^{34_3}$	$GM^{34_3}(0,2)$	$\mu_x^{34_3} = \exp\left(\beta_1^{34_3} + \beta_2^{34_3}x\right)$	$\beta_1^{34_3}$ and $\beta_2^{34_3}$
	$GM^{34_3}(1,2)$	$\mu_x^{34_3} = \alpha^{34_3} + \exp\left(\beta_3^{34_3} + \beta_4^{34_3}x\right)$	$\alpha^{34_3}, \beta_3^{34_3}$ and $\beta_4^{34_3}$
$\mu_x^{15}, \mu_x^{25}, \mu_x^{35}$	$GM^{i5}(0,2)$	$\mu_x^{i5} = \exp\left(\beta_1^{i5} + \beta_2^{i5}x\right)$	β_1^{i5} and β_2^{i5}
	$GM^{i5}(1,2)$	$\mu_x^{i5} = \alpha^{i5} + \exp\left(\beta_3^{i5} + \beta_4^{i5}x\right)$	$\alpha^{i5}, \beta_3^{i5}$ and β_4^{i5}

We shall show that these assumptions are fit in the later sections.

Transition probabilities estimation

This section provides the derivation of formula of the transition probabilities from healthy (State 0) to critical illness (State *i*) for $i = 1,2,3$.

Using the assumptions of the transition intensities μ_x^{0i} and μ_x^{i5} shown in Table 1, and considering equation (6), the probability of an insured currently aged *x* and is in State 0 will remain in the same state until time *t* is:

$$ {}_tp_x^{\overline{00}} = \exp\left(-\int_0^t \sigma_k^{01} + \sigma_k^{02} + \sigma_k^{03} + \alpha^{05} + e^{\beta_1^{05}+\beta_2^{05}(x+u)}\, du\right) $$

for $k = 0,1,2,\ldots,n-1$, $x_k \le x < x_{k+1}$ and $t \le x_{k+1} - x$ and for $k = n$ where $x > x_n$ and $\forall t$.

Fixed $\dot{\beta}_1^{05} = \exp(\beta_1^{05})$, the solution of the previous equation is:

$$ {}_tp_x^{\overline{00}} = \exp\left[-t(\sigma_k^{01} + \sigma_k^{02} + \sigma_k^{03} + \alpha^{05}) - \frac{\dot{\beta}_1^{05}}{\beta_2^{05}}\left(e^{\beta_2^{05}(x+t)} - e^{\beta_2^{05}(x)}\right)\right] \tag{22} $$

for $k = 0,1,2,\ldots,n-1$, $x_k \le x < x_{k+1}$ and $t \le x_{k+1} - x$ and for $k = n$ where $x > x_n$ and $\forall t$.

By using equations (7), and the assumption of the transition intensities $\mu_x^{i4_i}$ and μ_x^{i5} shown in Table 1, for all $i = 1,2,3$ and $\forall t$, we derive the probability to remain in State *i* until time *t*, where $\dot{\beta}_1^{i4_i} = \exp(\beta_1^{i4_i})$:

$$ \begin{aligned} {}_tp_x^{\bar{ii}} &= \exp\left(-\int_0^t \mu_{x+u}^{i4_i} + \mu_{x+u}^{i5}\, du\right) \\ &= \exp\left(-\int_0^t \alpha^{i4_i} + e^{\beta_1^{i4_i}+\beta_2^{i4_i}(x+u)} + \alpha^{i5} + e^{\beta_1^{i5}+\beta_2^{i5}(x+u)}\, du\right) \\ &= \exp\left[-t\left(\alpha^{i4_i} + \alpha^{05}\right) - \frac{\dot{\beta}_1^{i4_i}}{\beta_2^{i4_i}}\left(e^{\beta_2^{i4_i}(x+t)} - e^{\beta_2^{i4_i}(x)}\right) - \frac{\dot{\beta}_1^{i5}}{\beta_2^{i5}}\left(e^{\beta_2^{i5}(x+t)} - e^{\beta_2^{i5}(x)}\right)\right] \end{aligned} \tag{23} $$

From equation (5), we use the equations (22) and (23) as well as considering the assumption of transition intensities μ_x^{0i} shown in Table 1, we have

$$ \begin{aligned} {}_tp_x^{0i} &= \int_0^t {}_tp_x^{\overline{00}}\, \mu_{x+u}^{0i} \cdot {}_{t-u}p_{x+u}^{\bar{ii}}\, du \\ &= \int_0^t \exp\left[-u\left(\sigma_k^{01} + \sigma_k^{02} + \sigma_k^{03} + \alpha^{05}\right) - \frac{\dot{\beta}_1^{05}}{\beta_2^{05}}\left(e^{\beta_2^{05}(x+u)} - e^{\beta_2^{05}(x)}\right)\right] \cdot \sigma_k^{0i} \\ &\quad \cdot \exp\left[-(t-u)\left(\alpha^{i4_i} + \alpha^{05}\right) - \frac{\dot{\beta}_1^{i4_i}}{\beta_2^{i4_i}}\left(e^{\beta_2^{i4_i}(x+t)} - e^{\beta_2^{i4_i}(x+u)}\right) - \frac{\dot{\beta}_1^{i5}}{\beta_2^{i5}}\left(e^{\beta_2^{i5}(x+t)} - e^{\beta_2^{i5}(x+u)}\right)\right] du \\ &= \sigma_k^{0i} \cdot \exp\left[\frac{\dot{\beta}_1^{05}}{\beta_2^{05}} e^{\beta_2^{05}(x)} - \frac{\dot{\beta}_1^{i5}}{\beta_2^{i5}} e^{\beta_2^{i5}(x+t)} - \frac{\dot{\beta}_1^{i4_i}}{\beta_2^{i4_i}} e^{\beta_2^{i4_i}(x+t)} - t\left(\alpha^{i4_i} + \alpha^{05}\right)\right] \\ &\quad \cdot \int_0^t \exp\left[-u\left(\sigma_k^{01} + \sigma_k^{02} + \sigma_k^{03} + \alpha^{05} - \alpha^{i4_i} - \alpha^{i5}\right) - \frac{\dot{\beta}_1^{05}}{\beta_2^{05}} e^{\beta_2^{05}(x+u)}\right] \\ &\quad \cdot \exp\left[\frac{\dot{\beta}_1^{i4_i}}{\beta_2^{i4_i}} e^{\beta_2^{i4_i}(x+u)} + \frac{\dot{\beta}_1^{i5}}{\beta_2^{i5}} e^{\beta_2^{i5}(x+u)}\right] du \end{aligned} \tag{24} $$

for $k = 0,1,2,\ldots,n-1$, $x_k \le x < x_{k+1}$ and $t \le x_{k+1} - x$ and for $k = n$ where $x > x_n$ and $\forall t$.

To solve equation (24), we assume the following an approximation solution according to a Taylor series expansion:

$$\exp\left(e^{\beta_2^{ij}u}\right) \cong \exp\left[e^{\beta_2^{ij}\frac{t}{2}} + e^{\beta_2^{ij}\frac{t}{2}}\beta_2^{ij}\left(u - \frac{t}{2}\right)\right]$$
$$= \exp\left[e^{\beta_2^{ij}\frac{t}{2}}\left(1 + \beta_2^{ij}\left(u - \frac{t}{2}\right)\right)\right]$$

for $i = 0,1,2,3$, $j = 4_1, 4_2, 4_3, 5$ and $i < j$.

Hence, we obtain

$$\exp\left[-u\left(\sigma_k^{01} + \sigma_k^{02} + \sigma_k^{03} + \alpha^{05} - \alpha^{i4_i} - \alpha^{i5}\right) - \frac{\dot{\beta}_1^{05}}{\beta_2^{05}}e^{\beta_2^{05}(x+u)} + \frac{\dot{\beta}_1^{i4_i}}{\beta_2^{i4_i}}e^{\beta_2^{i4_i}(x+u)} + \frac{\dot{\beta}_1^{i5}}{\beta_2^{i5}}e^{\beta_2^{i5}(x+u)}\right]$$
$$= \exp\left[-u\left(\sigma_k^{01} + \sigma_k^{02} + \sigma_k^{03} + \alpha^{05} - \alpha^{i4_i} - \alpha^{i5}\right) - \frac{\dot{\beta}_1^{05}}{\beta_2^{05}}e^{\beta_2^{05}(x)}\cdot e^{\beta_2^{05}(u)} + \frac{\dot{\beta}_1^{i4_i}}{\beta_2^{i4_i}}e^{\beta_2^{i4_i}(x)}\cdot e^{\beta_2^{i4_i}(u)}\right]$$
$$\cdot \exp\left[\frac{\dot{\beta}_1^{i5}}{\beta_2^{i5}}e^{\beta_2^{i5}(x)}\cdot e^{\beta_2^{i5}(u)}\right]$$
$$= \exp\left[-u\left(\sigma_k^{01} + \sigma_k^{02} + \sigma_k^{03} + \alpha^{05} - \alpha^{i4_i} - \alpha^{i5}\right) - \frac{\dot{\beta}_1^{05}}{\beta_2^{05}}e^{\beta_2^{05}(x)}\cdot e^{\beta_2^{05}\frac{t}{2}}\left(1 + \beta_2^{05}\left(u - \frac{t}{2}\right)\right)\right]$$
$$\cdot \exp\left[\frac{\dot{\beta}^{i4_i}}{\beta_2^{i4_i}}e^{\beta_2^{i4_i}(x)}\cdot e^{\beta_2^{i4_i}\frac{t}{2}}\left(1 + \beta_2^{i4_i}\left(u - \frac{t}{2}\right)\right) + \frac{\dot{\beta}_1^{i5}}{\beta_2^{i5}}e^{\beta_2^{i5}(x)}\cdot e^{\beta_2^{i5}\frac{t}{2}}\left(1 + \beta_2^{i5}\left(u - \frac{t}{2}\right)\right)\right]$$
$$= \exp\left[-u\left(\sigma_k^{01} + \sigma_k^{02} + \sigma_k^{03} + \alpha^{05} - \alpha^{i4_i} - \alpha^{i5}\right)\right]\cdot \exp\left[-\frac{\dot{\beta}_1^{05}}{\beta_2^{05}}e^{\beta_2^{05}\left(x+\frac{t}{2}\right)}\left(1 - \beta_2^{05}\frac{t}{2}\right)\right]$$
$$\cdot \exp\left[\frac{\dot{\beta}_1^{i4_i}}{\beta_2^{i4_i}}e^{\beta_2^{i4_i}\left(x+\frac{t}{2}\right)}\left(1 - \beta_2^{i4_i}\frac{t}{2}\right)\right]\cdot \exp\left[\frac{\dot{\beta}_1^{i5}}{\beta_2^{i5}}e^{\beta_2^{i5}\left(x+\frac{t}{2}\right)}\left(1 - \beta_2^{i5}\frac{t}{2}\right)\right]$$
$$\cdot \exp\left[-\frac{\dot{\beta}_1^{05}}{\beta_2^{05}}e^{\beta_2^{05}\left(x+\frac{t}{2}\right)}\left(\beta_2^{05}u\right)\right]\exp\left[\frac{\dot{\beta}_1^{i4_i}}{\beta_2^{i4_i}}e^{\beta_2^{i4_i}\left(x+\frac{t}{2}\right)}\left(\beta_2^{i4_i}u\right)\right]\cdot \exp\left[\frac{\dot{\beta}_1^{i5}}{\beta_2^{i5}}e^{\beta_2^{i5}\left(x+\frac{t}{2}\right)}\left(\beta_2^{i5}u\right)\right]$$
$$= \exp\left[u\left(-\sigma_k^{01} - \sigma_k^{02} - \sigma_k^{03} - \alpha^{05} + \alpha^{i4_i} + \alpha^{i5} - \dot{\beta}_1^{05}e^{\beta_2^{05}\left(x+\frac{t}{2}\right)} + \dot{\beta}_1^{i4_i}e^{\beta_2^{i4_i}\left(x+\frac{t}{2}\right)} + \dot{\beta}_1^{i5}e^{\beta_2^{i5}\left(x+\frac{t}{2}\right)}\right)\right]$$
$$\cdot \exp\left[-\frac{\dot{\beta}_1^{05}}{\beta_2^{05}}e^{\beta_2^{05}\left(x+\frac{t}{2}\right)}\left(1 - \beta_2^{05}\frac{t}{2}\right) + \frac{\dot{\beta}_1^{i4_i}}{\beta_2^{i4_i}}e^{\beta_2^{i4_i}\left(x+\frac{t}{2}\right)}\left(1 - \beta_2^{i4_i}\frac{t}{2}\right) + \frac{\dot{\beta}_1^{i5}}{\beta_2^{i5}}e^{\beta_2^{i5}\left(x+\frac{t}{2}\right)}\left(1 - \beta_2^{i5}\frac{t}{2}\right)\right]. \quad (25)$$

By using (25), equation (24) becomes

$$
\begin{aligned}
{}_tp_x^{0i} &= \sigma_k^{0i} \cdot \exp\left[\frac{\dot\beta_1^{05}}{\beta_2^{05}} e^{\beta_2^{05}(x)} - \frac{\dot\beta_1^{i5}}{\beta_2^{i5}} e^{\beta_2^{i5}(x+t)} - \frac{\dot\beta_1^{i4_i}}{\beta_2^{i4_i}} e^{\beta_2^{i4_i}(x+t)} - t\left(\alpha^{i4_i} + \alpha^{05}\right)\right] \\
&\quad \cdot \int_0^t \exp\left[-u\left(\sigma_k^{01} + \sigma_k^{02} + \sigma_k^{03} + \alpha^{05} - \alpha^{i4_i} - \alpha^{i5}\right) - \frac{\dot\beta_1^{05}}{\beta_2^{05}} e^{\beta_2^{05}(x+u)}\right] \\
&\quad \cdot \exp\left[\frac{\dot\beta_1^{i4_i}}{\beta_2^{i4_i}} e^{\beta_2^{i4_i}(x+u)} + \frac{\dot\beta_1^{i5}}{\beta_2^{i5}} e^{\beta_2^{i5}(x+u)}\right] du \\
&= \sigma_k^{0i} \cdot \exp\left[\frac{\dot\beta_1^{05}}{\beta_2^{05}} e^{\beta_2^{05}(x)} - \frac{\dot\beta_1^{i5}}{\beta_2^{i5}} e^{\beta_2^{i5}(x+t)} - \frac{\dot\beta_1^{i4_i}}{\beta_2^{i4_i}} e^{\beta_2^{i4_i}(x+t)} - t\left(\alpha^{i4_i} + \alpha^{05}\right)\right] \\
&\quad \cdot \int_0^t \exp\left[u\left(-\sigma_k^{01} - \sigma_k^{02} - \sigma_k^{03} - \alpha^{05} + \alpha^{i4_i} + \alpha^{i5} - \dot\beta_1^{05} e^{\beta_2^{05}\left(x+\frac{t}{2}\right)} + \dot\beta_1^{i4_i} e^{\beta_2^{i4_i}\left(x+\frac{t}{2}\right)} + \dot\beta_1^{i5} e^{\beta_2^{i5}\left(x+\frac{t}{2}\right)}\right)\right] \\
&\quad \cdot \exp\left[-\frac{\dot\beta_1^{05}}{\beta_2^{05}} e^{\beta_2^{05}\left(x+\frac{t}{2}\right)}\left(1-\beta_2^{05}\frac{t}{2}\right) + \frac{\dot\beta_1^{i4_i}}{\beta_2^{i4_i}} e^{\beta_2^{i4_i}\left(x+\frac{t}{2}\right)}\left(1-\beta_2^{i4_i}\frac{t}{2}\right) + \frac{\dot\beta_1^{i5}}{\beta_2^{i5}} e^{\beta_2^{i5}\left(x+\frac{t}{2}\right)}\left(1-\beta_2^{i5}\frac{t}{2}\right)\right] du \\
&= \sigma_k^{0i} \cdot \exp\left[\frac{\dot\beta_1^{05}}{\beta_2^{05}} e^{\beta_2^{05}(x)} - \frac{\dot\beta_1^{i5}}{\beta_2^{i5}} e^{\beta_2^{i5}(x+t)} - \frac{\dot\beta_1^{i4_i}}{\beta_2^{i4_i}} e^{\beta_2^{i4_i}(x+t)} - t\left(\alpha^{i4_i} + \alpha^{05}\right)\right] \\
&\quad \cdot \exp\left[-\frac{\dot\beta_1^{05}}{\beta_2^{05}} e^{\beta_2^{05}\left(x+\frac{t}{2}\right)}\left(1-\beta_2^{05}\frac{t}{2}\right) + \frac{\dot\beta_1^{i4_i}}{\beta_2^{i4_i}} e^{\beta_2^{i4_i}\left(x+\frac{t}{2}\right)}\left(1-\beta_2^{i4_i}\frac{t}{2}\right) + \frac{\dot\beta_1^{i5}}{\beta_2^{i5}} e^{\beta_2^{i5}\left(x+\frac{t}{2}\right)}\left(1-\beta_2^{i5}\frac{t}{2}\right)\right] \\
&\quad \cdot \int_0^t \exp\left[u\left(-\sigma_k^{01} - \sigma_k^{02} - \sigma_k^{03} - \alpha^{05} + \alpha^{i4_i} + \alpha^{i5} - \dot\beta_1^{05} e^{\beta_2^{05}\left(x+\frac{t}{2}\right)} + \dot\beta_1^{i4_i} e^{\beta_2^{i4_i}\left(x+\frac{t}{2}\right)} + \dot\beta_1^{i5} e^{\beta_2^{i5}\left(x+\frac{t}{2}\right)}\right)\right] \\
&\quad \cdot \exp\left[u\left(\dot\beta_1^{i5} e^{\beta_2^{i5}\left(x+\frac{t}{2}\right)}\right)\right] du \\
&= \sigma_k^{0i} \cdot \exp\left[\frac{\dot\beta_1^{05}}{\beta_2^{05}} e^{\beta_2^{05}(x)} - \frac{\dot\beta_1^{i5}}{\beta_2^{i5}} e^{\beta_2^{i5}(x+t)} - \frac{\dot\beta_1^{i4_i}}{\beta_2^{i4_i}} e^{\beta_2^{i4_i}(x+t)} - t\left(\alpha^{i4_i} + \alpha^{05}\right)\right] \\
&\quad \cdot \exp\left[-\frac{\dot\beta_1^{05}}{\beta_2^{05}} e^{\beta_2^{05}\left(x+\frac{t}{2}\right)}\left(1-\beta_2^{05}\frac{t}{2}\right) + \frac{\dot\beta_1^{i4_i}}{\beta_2^{i4_i}} e^{\beta_2^{i4_i}\left(x+\frac{t}{2}\right)}\left(1-\beta_2^{i4_i}\frac{t}{2}\right) + \frac{\dot\beta_1^{i5}}{\beta_2^{i5}} e^{\beta_2^{i5}\left(x+\frac{t}{2}\right)}\left(1-\beta_2^{i5}\frac{t}{2}\right)\right] \\
&\quad \cdot \frac{\exp\left[t\left(-\sigma_k^{01} - \sigma_k^{02} - \sigma_k^{03} - \alpha^{05} - \dot\beta_1^{05} e^{\beta_2^{05}\left(x+\frac{t}{2}\right)} + \zeta_i\right)\right] - 1}{-\sigma_k^{01} - \sigma_k^{02} - \sigma_k^{03} - \alpha^{05} - \dot\beta_1^{05} e^{\beta_2^{05}\left(x+\frac{t}{2}\right)} + \zeta_i},
\end{aligned}
\tag{26}
$$

where $\zeta_i = \alpha^{i4_i} + \alpha^{i5} + \dot\beta_1^{i4_i} e^{\beta_2^{i4_i}\left(x+\frac{t}{2}\right)} + \dot\beta_1^{i5} e^{\beta_2^{i5}\left(x+\frac{t}{2}\right)}$.

Equations (19) and (20) are possible to use to obtain an estimation of the unknown parameters, σ_k^{0i} for all $k = 0,1,\ldots,n-1$ with an iterative approach starting from the initial age group $[x_k, x_{k+1})$, provided that both equations (22) and (26) are substituted when prevalence rates $f_{x+s}^{(i)}$ are available.

Data

We will be using the Canadian health data in our study. We describe the Canadian health data collected from Public Health Infobase provided by *CCDSS* [5] as follows. The data for stroke and heart attack (acute myocardial infraction) in year 2015 is reported by the hospital from the patients who stayed in the hospital for a certain time period (*CCDSS* [5]). The prevalence rates of stroke and heart attack presented in Table 2 are the accumulation of number of cases reported based on the yearly basis. The collected data for the prevalence rates of stroke and heart attack are categorised into 5 age groups for male and female, i.e., 20 − 34, 35 − 49, 50 – 64, 65 – 79 and 80+. Based on Table 2, the prevalence rates increase as age increases for both genders and both critical illnesses. At younger age, males have lower prevalence rates of stroke as compare to females. However, starting age 50 onwards, the prevalence rates of stroke for males are higher than females. As for the heart attack, males has higher prevalence rates than females for all ages.

Table 2. Prevalence rates of stroke and heart attack in year 2015

Prevalence Rates				
Age group	Stroke		Heart attack	
	Males	Females	Males	Females
20 – 34	0.0012	0.0014	0.0003	0.0001
35 – 49	0.0057	0.0064	0.0055	0.0015
50 – 64	0.0240	0.0201	0.0351	0.0098
65 – 79	0.0773	0.0596	0.0849	0.0312
80+	0.1847	0.1643	0.1349	0.0799

On the other hand, the prevalence rates of cancer (malignant neoplasm) in year 2015 are categorised into 8 age groups for males and females, is obtained from [26] which is presented in Table 3. At younger age between age 20 to age 59, the prevalence rates of cancer for males are lower than females. Starting age 60 onwards, males have higher prevalence rates of cancer as compared to females. In general, males tend to have higher prevalence rates of stroke, heart attack and cancer at older age, based on Table 2 and Table 3.

Table 3. Prevalence rates of cancer in year 2015

Prevalence Rates		
Age group	Cancer	
	Males	Females
20 – 29	0.0016	0.0018
30 – 39	0.0029	0.0053
40 – 49	0.0060	0.0126
50 – 59	0.0177	0.0241
60 – 69	0.0480	0.0397
70 – 79	0.0835	0.0560
80 – 89	0.0919	0.0565
90+	0.0757	0.0431

The number of populations in year 2015 for males and females obtained from [27], are presented in Table 4, where it is divided into 15 age groups, starting 20 − 24, 25 − 29, …, 85 − 89 and 90+. Besides, the number of deaths for stroke, heart attack, cancer and all causes other than these 3 critical illnesses in year 2015 are obtained from [28] and shown in Table 5. Similar to Table 4, the data in Table 5 are categorised based on 2 genders and 15 age groups.

Table 4. Number of populations in year 2015

Number of populations		
Age group	Males	Females
20 – 24	1244697	1150926
25 – 29	1239356	1190201
30 – 34	1230618	1229883
35 – 39	1174086	1197143
40 – 44	1167211	1182711
45 – 49	1220275	1225541
50 – 54	1392935	1390415
55 – 59	1300456	1314212
60 – 64	1102960	1140251
65 – 69	926287	976717
70 – 74	649566	708146
75 – 79	452282	530742
80 – 84	317644	417363
85 – 89	177089	290076
90+	77062	191352

Table 5. Number of deaths due to critical illnesses in year 2015

Number of deaths								
Age group	Stoke		Heart attack		Cancer		All causes	
	Males	Females	Males	Females	Males	Females	Males	Females
20 – 24	1	5	0	0	60	31	897	376
25 – 29	5	7	2	1	58	76	1061	441
30 – 34	14	9	7	1	126	148	1131	577
35 – 39	18	14	24	8	176	232	1238	698
40 – 44	42	32	54	14	291	421	1735	1094
45 – 49	58	48	147	31	618	764	2727	1667
50 – 54	118	106	306	64	1471	1660	4887	3326
55 – 59	190	158	539	130	2725	2725	7456	4921
60 – 64	249	191	710	259	4188	3589	10021	6707
65 – 69	382	329	890	339	5402	4372	12788	8708
70 – 74	563	462	928	451	5893	4781	14242	10447
75 – 79	751	735	913	629	5995	4771	16341	12918
80 – 84	1148	1321	1204	927	6034	5096	20196	18664
85 – 89	1199	1841	1208	1242	4519	4257	20006	23499
90+	1019	2768	994	1939	2747	3631	16940	35512

We will use the data obtained from Tables 2 – 5 to estimate the transition intensities μ^{ij} in the later sections.

Empirical Analysis

By using the collected data presented above, we perform the estimation of transition intensities in this section. As discussed in previous sections, selected equations and mortality models will be applied to the Canadian health data in Tables 2 – 5 to estimate the required transition intensities. A comparison of

mortality models and analysis of transition intensities will be discussed and presented in details under this section.

Estimation of Transition Intensity $\mu_x^{i4_i}$

By using Tables 2 – 5, we estimate the transition intensities with the defined mathematical formula in the sections "Transition intensities framework" and "Transition probabilities estimation". Based on the collected data as in Tables 4 and 5, the values of transition intensities for $\mu_x^{14_1}$, $\mu_x^{24_2}$ and $\mu_x^{34_3}$ are calculated discretely using equation (2) as presented in Table 6.

Table 6. Discrete transition intensities $\mu_x^{i4_i}$ for $i = 1,2,3$

Age group	Transition intensities					
	$\mu_x^{14_1}$		$\mu_x^{24_2}$		$\mu_x^{34_3}$	
	Males	Females	Males	Females	Males	Females
20 – 24	0.00067	0.00310	0.00000	0.00000	0.03030	0.01515
25 – 29	0.00336	0.00420	0.00538	0.00840	0.02941	0.03592
30 – 34	0.00948	0.00523	0.01897	0.00813	0.03492	0.02292
35 – 39	0.00269	0.00183	0.00372	0.00445	0.05113	0.03691
40 – 44	0.00631	0.00423	0.00841	0.00789	0.04161	0.02831
45 – 49	0.00834	0.00612	0.02190	0.01687	0.08453	0.04957
50 – 54	0.00353	0.00379	0.00626	0.00470	0.05959	0.04950
55 – 59	0.00609	0.00598	0.01181	0.01009	0.11824	0.08598
60 – 64	0.00941	0.00833	0.01834	0.02318	0.07908	0.07920
65 – 69	0.00534	0.00565	0.01132	0.01112	0.12146	0.11263
70 – 74	0.01121	0.01095	0.01683	0.02041	0.10868	0.12064
75 – 79	0.02148	0.02324	0.02378	0.03799	0.15879	0.16063
80 – 84	0.01957	0.01926	0.02810	0.02780	0.20663	0.21618
85 – 89	0.03666	0.03863	0.05057	0.05359	0.27756	0.25983
90+	0.07159	0.08804	0.09561	0.12682	0.47102	0.44012

The discrete transition intensities $\mu_x^{14_1}$, $\mu_x^{24_2}$ and $\mu_x^{34_3}$ of the insureds currently aged *x* are fitted with the proposed mortality models shown in Table 1. The fitted transition intensities under both Gompertz and Makeham mortality models for males and females are presented in Figures 2 – 7.

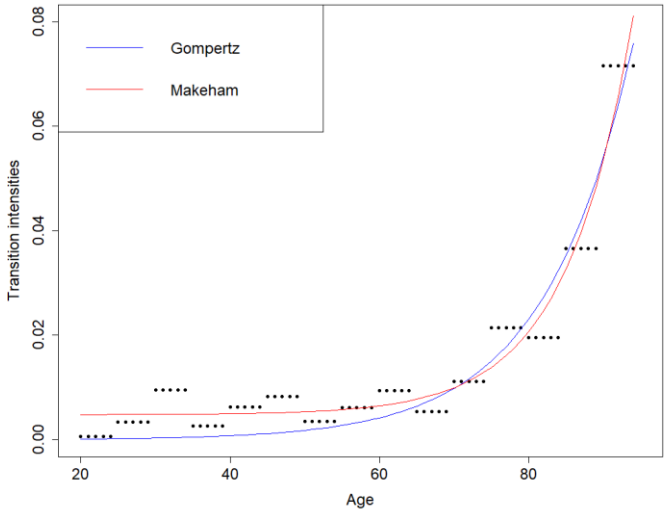

Figure 2. Gompertz and Makeham mortality models of $\mu_x^{14_1}$ (Stroke) on male

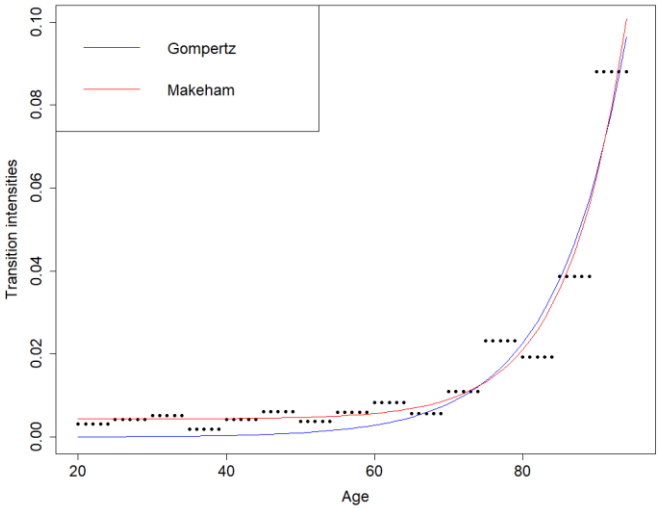

Figure 3. Gompertz and Makeham mortality models of $\mu_x^{14_1}$ (Stroke) on female

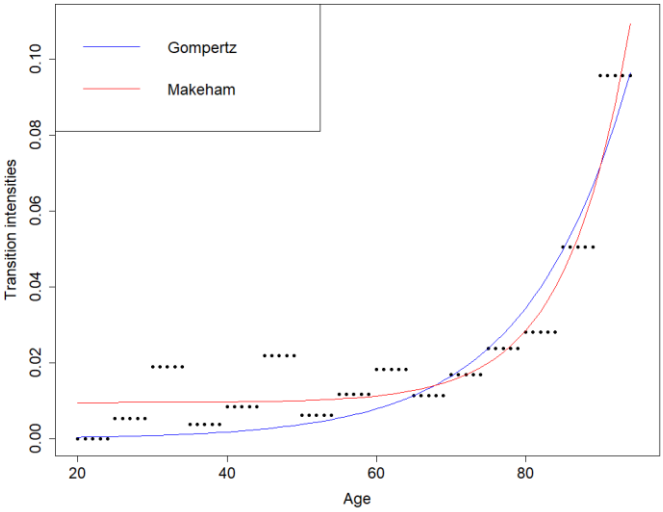

Figure 4. Gompertz and Makeham mortality models of $\mu_x^{24_2}$ (Heart Attack) on male

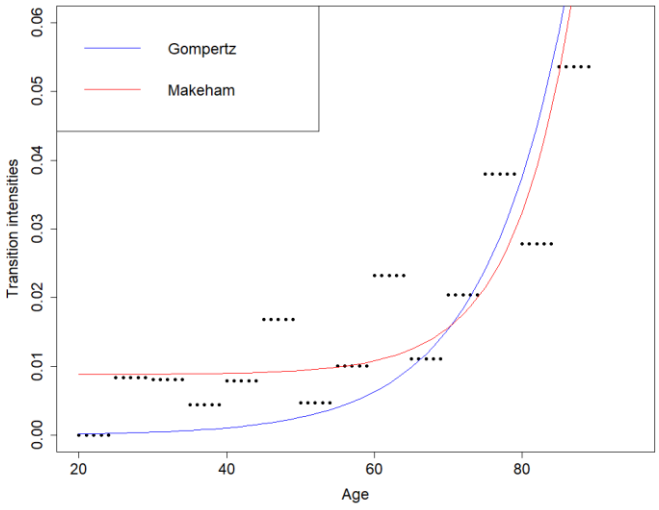

Figure 5. Gompertz and Makeham mortality models of $\mu_x^{24_2}$ (Heart Attack) on female

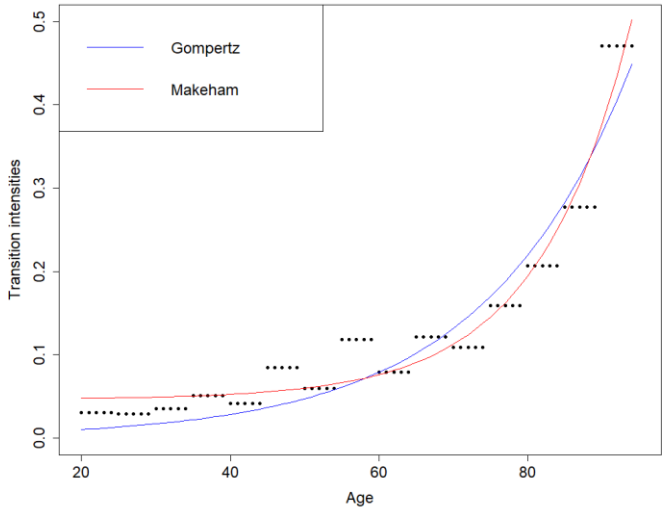

Figure 6. Gompertz and Makeham mortality models of $\mu_x^{34_3}$ (Cancer) on male

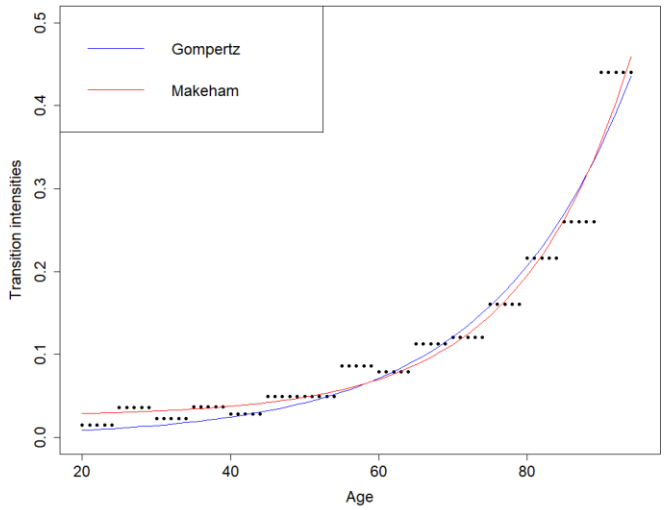

Figure 7. Gompertz and Makeham mortality models of $\mu_x^{34_3}$ (Cancer) on female

Figures 2 – 7 show that the Gompertz mortality model of $\mu_x^{14_1}$ and $\mu_x^{24_2}$ for both males and females are underestimated during the younger age and work well after age 70. Conversely, Makeham mortality model of $\mu_x^{14_1}$ and $\mu_x^{24_2}$ on males and females are ideally fitted although there are a huge fluctuation in male data at the beginning of younger ages. Moreover, Gompertz mortality model of $\mu_x^{34_3}$ is slightly underestimated on males. For the transition intensity of $\mu_x^{34_3}$ under female, both Gompertz and Makeham mortality model provide a similar curve which are closer to the observed transition intensities. By observing the graphs of both Gompertz and Makeham mortality models, it is hard to differentiate which models are preferred in this study. Thus, a model selection is required for a better model adequacy check.

In this paper, we will consider the residual sum of square and residual standard error as the criteria of our model selection. Residual sum of square defines the variation of data which can be explained by the estimated model and residual standard error defines the standard error of the estimated model. According to Montgomery *et al*. [23], the mathematical formula of residual sum of square, denoted by RSS and residual standard error, denoted by RSE are:

$$RSS = \sum_{i=1}^{n} \left(y_i - \hat{y}_i \right)^2$$

and

$$RSE = \sqrt{\frac{RSS}{n-2}}.$$

According to the fundamental concept of model selection, a preferred model must be selected when the criterion of the goodness of fit are met. Thus, the model will be considered as a best fitting model if the residual sum of square and residual standard error of the model are smallest in values. We perform the model selection of the Gompertz and Makeham mortality models on $\mu_x^{i4_i}$ for $i = 1,2,3$. The results are then provided in Table 7.

Table 7. Model residual of Gompertz and Makeham mortality models of $\mu_x^{i4_i}$ for $i = 1,2,3$

Transition intensity	Mortality model	Gender	Model residual	
			Sum of square	Standard error
$\mu_x^{14_1}$ (Stroke)	Gompertz	Male	0.00235	0.00567
		Female	0.00311	0.00653
	Makeham	Male	0.00156	0.00465
		Female	0.00236	0.00573
$\mu_x^{24_2}$ (Heart Attack)	Gompertz	Male	0.00686	0.00969
		Female	0.00976	0.01156
	Makeham	Male	0.00391	0.00737
		Female	0.00704	0.00989
$\mu_x^{34_3}$ (Cancer)	Gompertz	Male	0.08840	0.0348
		Female	0.04260	0.02417
	Makeham	Male	0.05390	0.02735
		Female	0.03300	0.02141

Based on the results in Table 7, Makeham mortality model is preferred for the transition intensities $\mu_x^{14_1}$, $\mu_x^{24_2}$ and $\mu_x^{34_3}$ due to its smallest values of residual sum of square and residual standard error. Therefore, the values of the parameter of Makeham mortality model on the $\mu_x^{i4_i}$ for $i = 1,2,3$ are presented in Table 8.

Table 8. Parameter values of Makeham mortality model of $\mu_x^{i4_i}$ for $i = 1,2,3$

Transition intensity	Gender	Parameter values for $\mu_x^{i4_i}$		
		α^{i4_i}	$\beta_1^{i4_i}$	$\beta_2^{i4_i}$
$\mu_x^{14_1}$ (Stroke)	Male	0.00480	−13.12000	0.11220
	Female	0.00435	−14.13000	0.12540
$\mu_x^{24_2}$ (Heart Attack)	Male	0.00954	−13.48168	0.11889
	Female	0.00883	−13.67462	0.12406
$\mu_x^{34_3}$ (Cancer)	Male	0.04709	−8.38730	0.08086
	Female	0.02586	−7.14590	0.06712

Estimation of Transition Intensity μ_x^{05}

We consider a set of values of extra mortality for critical illnesses *i* (i.e. Stroke, Heart attack and Cancer), is denoted by γ_i, that is, $\gamma_i = 0$, $\gamma_i = \{0,2,4,6,7,8,9\}$ for $i = 1,2,3$ to observe the sensitivity of both Gompertz and Makeham mortality model to fit μ_x^{05}. The extra mortality will affect the transition intensity as it varies when we refer to equation (12). For example, an extra mortality of zero, $\gamma_i = 0$, reflects that there is no increasing in the number of lives in State *i*, $l_x^{(i)}$. When the extra mortality increases, $\gamma_i > 0$, it reflects that there are more people who diagnosed critical illnesses *i* than the observed number of lives in State *i*, $l_x^{(i)}$.

We calculate μ_x^{05} for each values of extra mortality using equation (12) presented in Table 9 and Table 10. Then, μ_x^{05} on each values of extra mortality will be fitted with the proposed mortality model (refer Table 1). The results for both Gompertz and Makeham mortality model under a set of values of γ_i are presented in Table 11.

Table 9. Discrete transition intensities μ_x^{05} when $\gamma_i \leq 6$

Transition intensities						
Age group	$\gamma_i = 0$		$\gamma_i = 3$		$\gamma_i = 6$	
	Males	Females	Males	Females	Males	Females
20 – 24	0.00067	0.00030	0.00067	0.00029	0.00066	0.00029
25 – 29	0.00080	0.00030	0.00080	0.00030	0.00079	0.00029
30 – 34	0.00080	0.00034	0.00079	0.00033	0.00078	0.00033
35 – 39	0.00087	0.00037	0.00083	0.00036	0.00080	0.00034
40 – 44	0.00115	0.00053	0.00110	0.00051	0.00105	0.00047
45 – 49	0.00156	0.00067	0.00148	0.00065	0.00141	0.00060
50 – 54	0.00215	0.00108	0.00175	0.00097	0.00147	0.00081
55 – 59	0.00308	0.00145	0.00250	0.00131	0.00211	0.00110
60 – 64	0.00442	0.00234	0.00334	0.00205	0.00269	0.00165
65 – 69	0.00660	0.00376	0.00405	0.00298	0.00292	0.00211
70 – 74	0.01056	0.00671	0.00608	0.00519	0.00427	0.00357
75 – 79	0.01920	0.01278	0.01105	0.00988	0.00776	0.00680
80 – 84	0.03718	0.02712	0.01664	0.01694	0.01072	0.00967
85 – 89	0.07386	0.05571	0.03305	0.03479	0.02129	0.01987
90+	0.15806	0.14201	0.07231	0.09019	0.04688	0.05214

Table 10. Discrete transition intensities μ_x^{05} when $\gamma_i > 6$

Transition intensities						
Age group	$\gamma_i = 7$		$\gamma_i = 8$		$\gamma_i = 9$	
	Males	Females	Males	Females	Males	Females
20 – 24	0.00066	0.00029	0.00066	0.00029	0.00065	0.00029
25 – 29	0.00079	0.00029	0.00078	0.00029	0.00078	0.00029
30 – 34	0.00078	0.00033	0.00077	0.00032	0.00077	0.00032
35 – 39	0.00079	0.00034	0.00078	0.00034	0.00077	0.00033

Transition intensities						
40 – 44	0.00103	0.00046	0.00102	0.00046	0.00100	0.00045
45 – 49	0.00139	0.00059	0.00137	0.00058	0.00135	0.00057
50 – 54	0.00140	0.00078	0.00133	0.00075	0.00127	0.00072
55 – 59	0.00200	0.00105	0.00191	0.00101	0.00182	0.00098
60 – 64	0.00253	0.00157	0.00238	0.00150	0.00225	0.00144
65 – 69	0.00267	0.00196	0.00246	0.00184	0.00228	0.00173
70 – 74	0.00388	0.00331	0.00356	0.00309	0.00329	0.00289
75 – 79	0.00706	0.00630	0.00647	0.00588	0.00598	0.00551
80 – 84	0.00958	0.00874	0.00866	0.00796	0.00790	0.00732
85 – 89	0.01903	0.01794	0.01721	0.01636	0.01570	0.01503
90+	0.04196	0.04716	0.03797	0.04305	0.03468	0.03960

Table 11. Model residual of Gompertz and Makeham mortality of μ_x^{05} with different values of extra mortality

Extra Mortality	Mortality model	Gender	Model residual	
			Sum of square	Standard error
$\gamma_i = 0$	Gompertz	Male	0.00565	0.00880
		Female	0.00565	0.00879
	Makeham	Male	0.00564	0.00885
		Female	0.00565	0.00886
$\gamma_i = 2$	Gompertz	Male	0.00181	0.00499
		Female	0.00231	0.00563
	Makeham	Male	0.00178	0.00497
		Female	0.00230	0.00566
$\gamma_i = 4$	Gompertz	Male	0.00091	0.00354
		Female	0.00126	0.00416
	Makeham	Male	0.00087	0.00347
		Female	0.00125	0.00417
$\gamma_i = 6$	Gompertz	Male	0.00056	0.00277
		Female	0.00080	0.00331
	Makeham	Male	0.00052	0.00268
		Female	0.00079	0.00331
$\gamma_i = 7$	Gompertz	Male	0.00046	0.00250
		Female	0.00066	0.00300
	Makeham	Male	0.00042	0.00240
		Female	0.00065	0.00300
$\gamma_i = 8$	Gompertz	Male	0.00038	0.00229
		Female	0.00055	0.00275
	Makeham	Male	0.00034	0.00218
		Female	0.00054	0.00274
$\gamma_i = 9$	Gompertz	Male	0.00033	0.00211
		Female	0.00047	0.00254
	Makeham	Male	0.00029	0.00199
		Female	0.00046	0.00253

We compare the results between both Gompertz and Makeham mortality models for the set of values of extra mortality γ_i in Table 11. The model which has the lowest value of both residual sum of square and residual standard error will fulfill the model selection criterion, hence, can be claimed as the best fitting model. Under $\gamma_i = 0$, the value of residual sum of square on both Gompertz and Makeham mortality models are too close to make decision on selecting the preferred model. However, for the residual standard error, Gompertz mortality model has smaller value compare to Makeham mortality model, for both males and females. Therefore, we cannot conclude that Gompertz mortality model is the best fitting model based on its residual sum of square and residual standard error since it does not fully fulfill the two model selection criterion.

Under $\gamma_i = 3$, Makeham mortality model is better fitted for male data. For female data, the model selection results does not fully fulfill the model selection criterion for neither Gompertz mortality model nor Makeham mortality model. The results for both male and female data are consistently similar for all $\gamma_i \leq 6$.

When we increase the value of γ_i to 7, we observed that Makeham mortality model has the best fitting for both male and female by having the smallest residual sum of square and residual error. The results fulfill the model selection criterion under $\gamma_i = 7$ and consistently similar throughout $\gamma_i = 8$ and $\gamma_i = 9$. Since the results of Makeham mortality model hold for $\gamma_i > 6$, we will select the minimum value of extra mortality as our final model, that is, $\gamma_i = 7$. Therefore, we can conclude that Makeham mortality model is preferred in the fitting of μ_x^{05} for both male and female when $\gamma_i > 6$. To strengthen the results on choosing the minimum values of the Makeham mortality model of $\gamma_i = 7$, we observe the μ_x^{05} from the graph as presented in Figures 8 – 9.

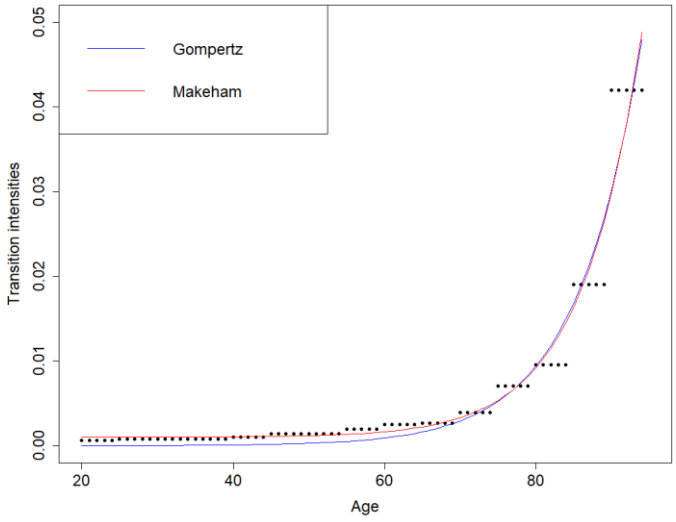

Figure 8. Gompertz and Makeham mortality models of μ_x^{05} on male when $\gamma_i = 7$

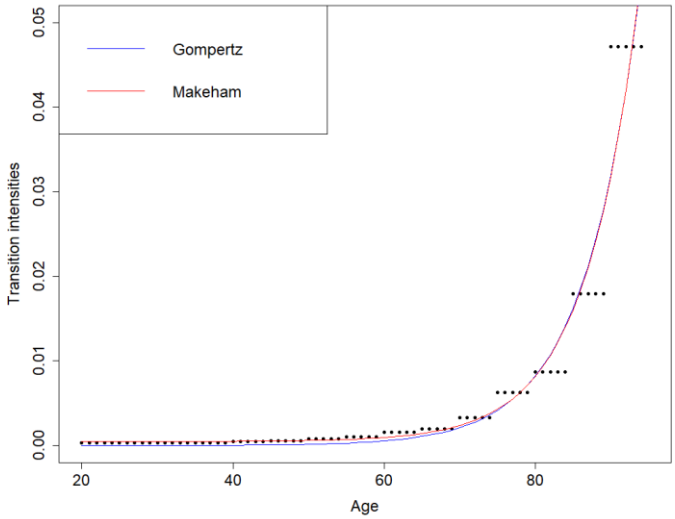

Figure 9. Gompertz and Makeham mortality models of μ_x^{05} on female when $\gamma_i = 7$

Based on Figures 8 – 9, Makeham mortality model provides a better fitting of μ_x^{05} on both genders for $\gamma_i = 7$ where all the mortality rates are ideally distributed throughout all ages. Conversely, Gompertz mortality model of μ_x^{05} on both genders does not provide a good fitting due to the underestimation of mortality rates before age 70. This evidence shows that Makeham mortality model is a better fitting model of μ_x^{05} for $\gamma_i = 7$. Since Makeham mortality model is preferred for both genders and for all transitions, therefore the values of the parameter of Makeham mortality model is provided in Table 12.

Table 12. Parameter values of Gompertz and Makeham mortality of μ_x^{05} with extra mortality $\gamma_i = 7$

Transition intensity	Gender	Parameter values for μ_x^{05}		
		α^{05}	β_1^{05}	β_2^{05}
μ_x^{05}	Male	0.00100	−14.92000	0.12630
	Female	0.00048	−16.11000	0.14060

Estimation of Transition Intensity μ_x^{i5}

Since μ_x^{05} follows a Makeham mortality model when $\gamma_i = 7$, we will estimate the transition intensities μ_x^{i5} for $i = 1,2,3$. First, we calculate μ_x^{i5} for $i = 1,2,3$ discretely using equation (15) and is presenetd in Table 13.

Note that the extra mortality γ_i is assumed to be the same throughout the three types of critical illnesses (Stroke, Heart attack and Cancer) when we first introduced it, therefore the values of μ_x^{i5} for $i = 1,2,3$ are equal using equation (15). After obtaining the discrete calculation of the values of μ_x^{i5} for $i = 1,2,3$, we fit the transition intensities with the proposed mortality models, that is, Gompertz and Makeham mortality model as shown in Table 1. Thus, the results of the model selection between the two mortality models, Gompertz and Makeham mortality models are presented in Table 14.

Table 13. Discrete transition intensities μ_x^{i5} for $i = 1,2,3$

Age group	Transition intensities					
	μ_x^{15}		μ_x^{25}		μ_x^{35}	
	Males	Females	Males	Females	Males	Females
20 – 24	0.00526	0.00231	0.00526	0.00231	0.00526	0.00231
25 – 29	0.00629	0.00235	0.00629	0.00235	0.00629	0.00235
30 – 34	0.00620	0.00260	0.00620	0.00260	0.00620	0.00260
35 – 39	0.00632	0.00272	0.00632	0.00272	0.00632	0.00272
40 – 44	0.00825	0.00371	0.00825	0.00371	0.00825	0.00371
45 – 49	0.01114	0.00470	0.01114	0.00470	0.01114	0.00470
50 – 54	0.01117	0.00625	0.01117	0.00625	0.01117	0.00625
55 – 59	0.01601	0.00843	0.01601	0.00843	0.01601	0.00843
60 – 64	0.02020	0.01258	0.02020	0.01258	0.02020	0.01258
65 – 69	0.02137	0.01570	0.02137	0.01570	0.02137	0.01570
70 – 74	0.03106	0.02649	0.03106	0.02649	0.03106	0.02649
75 – 79	0.05646	0.05043	0.05646	0.05043	0.05646	0.05043
80 – 84	0.07665	0.06989	0.07665	0.06989	0.07665	0.06989
85 – 89	0.15226	0.14354	0.15226	0.14354	0.15226	0.14354
90+	0.33567	0.37729	0.33567	0.37729	0.33567	0.37729

Table 14. Model residual of Gompertz and Makeham mortality models of μ_x^{i5} for $i = 1,2,3$

Transition intensity	Mortality model	Gender	Model residual	
			Sum of square	Standard error
μ_x^{i5}	Gompertz	Male	0.02930	0.02003
		Female	0.04220	0.02403
	Makeham	Male	0.02660	0.01921
		Female	0.04150	0.02400

Based on the results in Table 14, it is clearly show that Makeham mortality model on both genders provides a better fitting in μ_x^{i5} for $i = 1,2,3$ based on its smallest residual sum of square and residual standard error. The values of parameter of Makeham mortality model are then provided in Table 15. Furthermore, the graphs of μ_x^{i5} for $i = 1,2,3$ on both genders are plotted in Figures 10 – 11. We observed that the mortality rates under Makeham mortality model is ideally distributed throughout all ages whereas the mortality rates under Gompertz mortality model is underestimated before age 70.

Table 15. Parameter values of Makeham mortality model of μ_x^{i5} for $i = 1,2,3$

Transition intensity	Gender	Parameter values for μ_x^{i5}		
		α^{i5}	β_1^{i5}	β_2^{i5}
μ_x^{i5}	Male	0.00802	−12.83632	0.12633
	Female	0.00387	−14.03323	0.14055

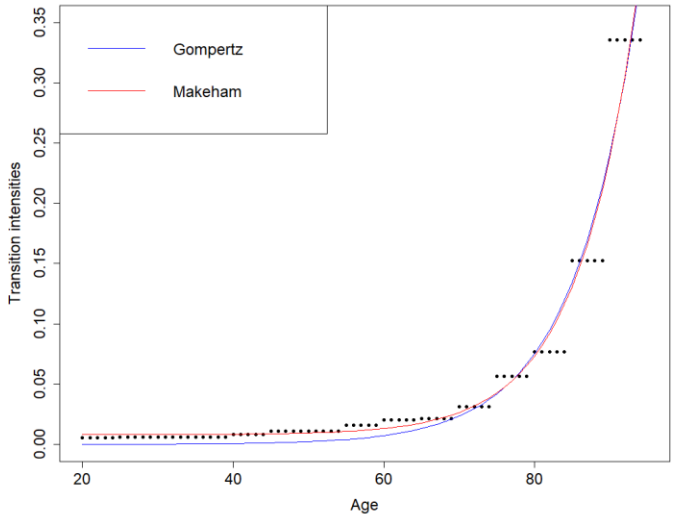

Figure 10. Gompertz and Makeham mortality models of μ_x^{i5} on male

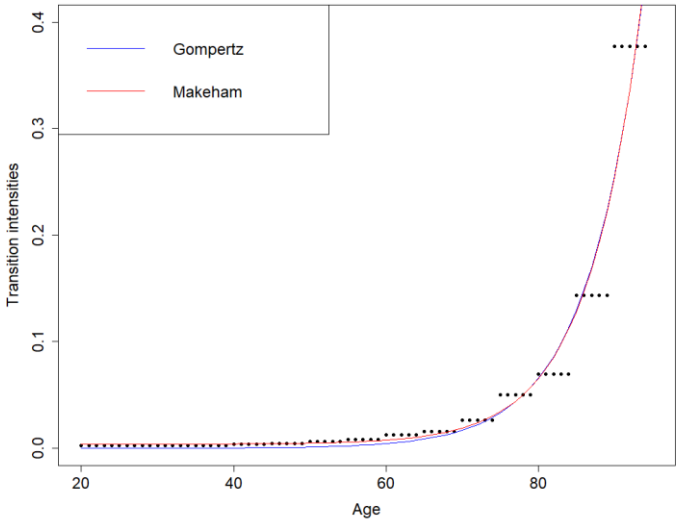

Figure 11. Gompertz and Makeham mortality models of μ_x^{i5} on female

Estimation of μ_x^{0i}

After obtaining the values of the parameter of Makeham mortality model for all transitions in the section "Empirical Analysis", we input the estimated values of the parameter of $\mu_x^{i4_i}$ for $i=1,2,3$ and μ_x^{05} into the equations (11), (14) and (17) when $\gamma_i=7$. Hence, equations (11) and (12) are applied to estimate the discrete values of μ_x^{0i} for $i=1,2,3$ by performing an iterative approach namely Newton-Raphson method. This iterative approach calculates a Jacobian of the function at each iteration. A quadratic convergence will be shown when the closed solution is iterated. With the use of this iterative approach, we obtained the numerical values of μ_x^{0i} for $i=1,2,3$ associated with the piecewise constant function shown in Table 1 is then presented in Table 16. We plotted the values of μ_x^{01}, μ_x^{02} and μ_x^{03} as presented in the Figures 12 – 17.

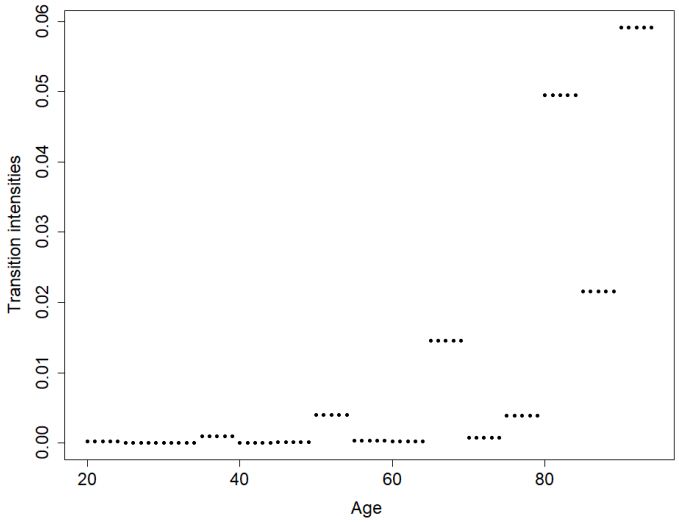

Figure 12. Gompertz and Makeham mortality models of μ_x^{01} on male

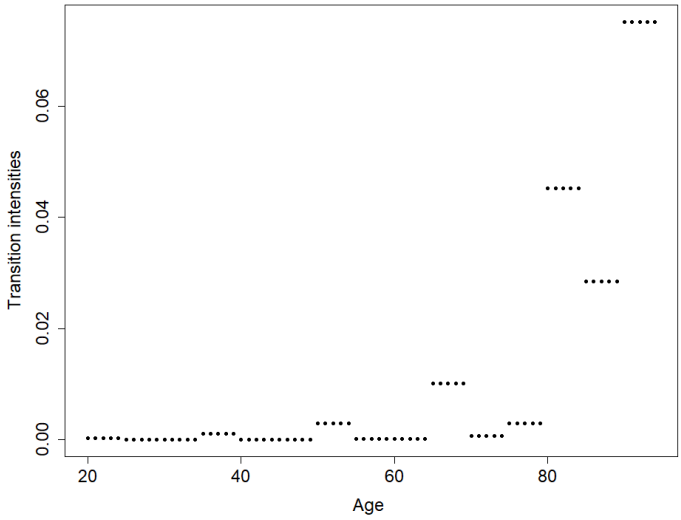

Figure 13. Gompertz and Makeham mortality models of μ_x^{01} on female

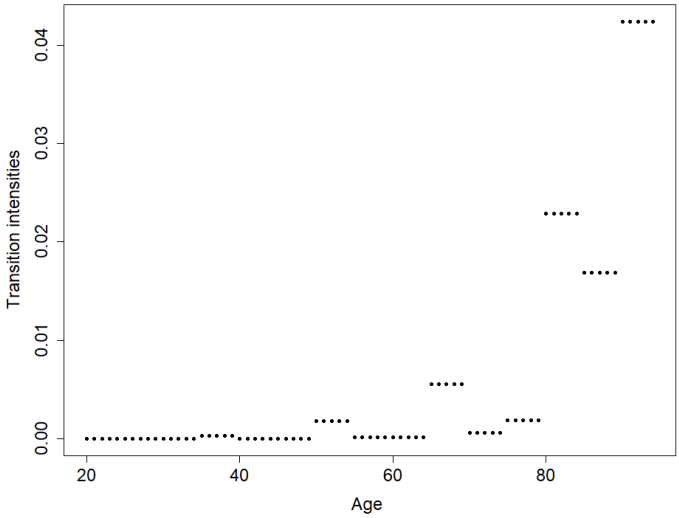

Figure 14. Gompertz and Makeham mortality models of μ_x^{02} on male

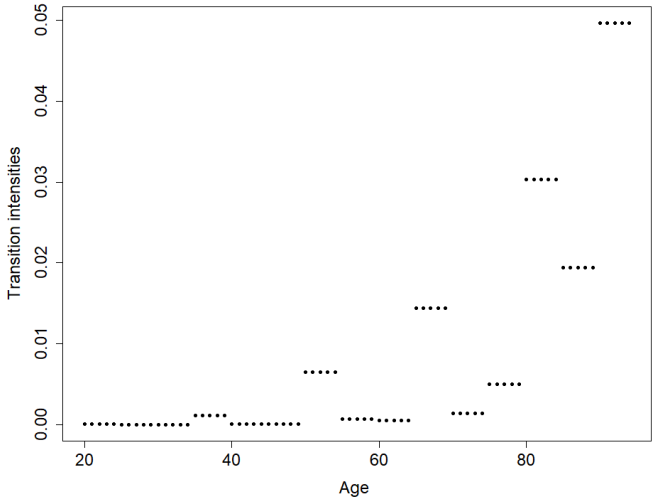

Figure 15. Gompertz and Makeham mortality models of μ_x^{02} on female

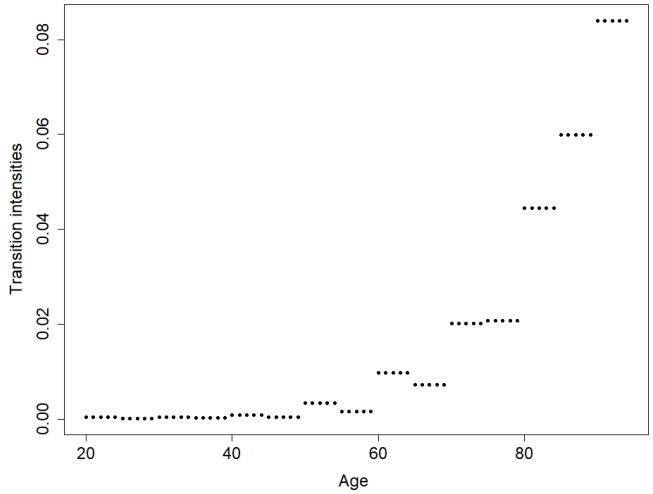

Figure 16. Gompertz and Makeham mortality models of μ_x^{03} on male

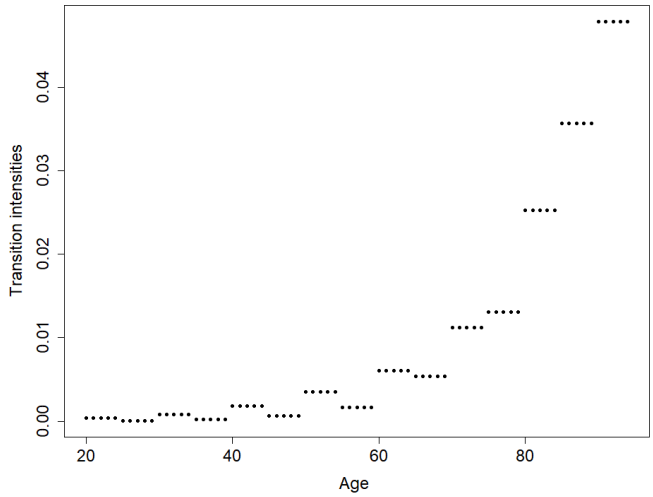

Figure 17. Gompertz and Makeham mortality models of μ_x^{03} on female

Table 16. Discrete transition intensities μ_x^{0i} for $i = 1,2,3$ with $\gamma_i = 7$

Age group	Transition intensities					
	μ_x^{01}		μ_x^{02}		μ_x^{03}	
	Males	Females	Males	Females	Males	Females
20 – 24	0.00025	0.00029	0.00006	0.00002	0.00037	0.00039
25 – 29	0.00001	0.00001	0.00000	0.00000	0.00009	0.00006
30 – 34	0.00001	0.00001	0.00001	0.00000	0.00040	0.00083
35 – 39	0.00095	0.00104	0.00110	0.00029	0.00017	0.00021
40 – 44	0.00007	0.00005	0.00009	0.00002	0.00091	0.00189
45 – 49	0.00007	0.00005	0.00010	0.00002	0.00040	0.00062
50 – 54	0.00405	0.00296	0.00660	0.00180	0.00339	0.00350
55 – 59	0.00036	0.00019	0.00071	0.00014	0.00152	0.00173
60 – 64	0.00041	0.00024	0.00080	0.00017	0.01023	0.00629
65 – 69	0.01389	0.00950	0.01363	0.00527	0.00646	0.00480
70 – 74	0.00231	0.00158	0.00316	0.00110	0.02206	0.01208
75 – 79	0.00374	0.00291	0.00489	0.00189	0.02165	0.01333
80 – 84	0.04427	0.03913	0.02624	0.01973	0.04003	0.02168
85 – 89	0.02777	0.03181	0.02387	0.01832	0.02387	0.03534
90+	0.06092	0.06976	0.05065	0.03929	0.08711	0.04411

Based on the graphs in Figures 12 – 17, μ_x^{01} and μ_x^{02} on both genders show an increasing trend along the age-group starting from age 20 to age 90 and above. A small fluctuation of values of μ_x^{01} and μ_x^{02} occurred before age 65 and a large fluctuation occurred after age 65. These fluctuations indicate that there is a slight increase in the prevalence rates of stroke and heart attack presented in Table 2 from age 20 to age 65 and increase after age 65.

Moreover, μ_x^{03} on both genders also show an increasing trend along the age group starting from age 20 to age 90 and above. On the male gender, the values of the transition intensities μ_x^{03} occur a small fluctuation before age 65 and a large fluctuation occurred after age 65. However, μ_x^{03} on female show a steady increasing trend and small fluctuation occurred along the age group from age 20 to age 90 and above. It indicates that there is an small increase in the prevalence rates of cancer presented in Table 3 from age 20 to age 89, and a slightly decrease for age 90 and above.

Now, we can conclude that the estimated transition intensities μ_x^{0i} for $i = 1,2,3$ are highly depending on the prevalence rates of the three types of critical illness using Newton-Raphson method.

Conclusion and Future Work

In this paper, we propose a multiple state model which consists of the three types of critical illness, i.e., stroke, heart attack and cancer, and two types of death, i.e., dead due to critical illness and dead due to other causes. The multiple state model is linked to the prevalence rate which may affect the number of deaths due to the three types of critical illness. When the prevalence rate of the critical illness increases, the number of death due to the three types of critical illness will increase along the number of populations. In this context, the consideration of transition intensities would be more suitable on the observation of the rate of occurrence of critical illness.

At the end of the study, Makeham mortality model of $\mu_x^{i4_i}$ for $i = 1,2,3$ is preferred to fit the Canadian health data collected due to its smallest residual sum of square and residual standard error. Similar

results have obtained for μ_x^{05} and μ_x^{i5} for with the assumption of $\gamma_i = 7$. We estimated μ_x^{0i} for $i = 1,2,3$ using Newton-Raphson method. Hence, the estimated μ_x^{0i} for $i = 1,2,3$ will be associated with the piecewise constant function.

The trend of $\mu_x^{i4_i}$, μ_x^{05} and μ_x^{i5} for $i = 1,2,3$ appears regularly with its time-dependent parameters. However, for μ_x^{01} and μ_x^{02}, the increasing trend fluctuated regularly from age 20 to age 65 and highly fluctuated after age 65 due to its increasing in prevalence rates of stroke and heart attack. μ_x^{03} on male provides a similar trend as μ_x^{01} and μ_x^{02}. Consequently, μ_x^{03} on female show a steady increasing trend along all ages due to the small amount of increase in percentage of the prevalence rates of cancer. In our next study, we will apply the estimated transition intensities to investigate effect of the insurance premiums in the critical illenss insurance by considering a few different types of insurance models.

Conflicts of Interest

The author(s) declare(s) that there is no conflict of interest regarding the publication of this paper.

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