MJFAS
Malaysian Journal of Fundamental and Applied Sciences

RESEARCH ARTICLE

Second Hankel Determinant of Bi-univalent Functions

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Abstract Let A be the class of functions which are analytic in the open unit disk $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ and having the form $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$. Denote S to be the class for all functions in A that are univalent in $\mathbb{D}$. Then, let σ denote the class of bi-univalent functions in $\mathbb{D}$. In this paper, we obtain the second Hankel determinant for certain classes of analytic bi-univalent function which are defined by subordinations in the open unit disk $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$. In particular, we determine the initial coefficients a_2, a_3 and a_4 and obtained the upper bound for the functional $|a_2a_4-a_3^2|$ of functions f in the classes of analytic bi-univalent function which are defined by subordinations in $\mathbb{D}$.

Keywords: Analytic functions, Bi-univalent functions, Second Hankel determinant, Subordination.

Introduction and Preliminaries

Let A be the class of analytic functions f is defined in the open unit disk $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ and normalized by the conditions $f(0)=0$ and $f'(0)=1$. Let S consists of all functions $f\in A$ such that f are univalent in $\mathbb{D}$. According to Brown and Churchill [4], a function f of the complex variable z is analytic at a point z_0 if it has a derivative at each point in some neighborhood of z_0. It follows that if f is analytic at a point z_0, it must be analytic at each point in some neighborhood of z_0. A function is analytic in an open set if it has a derivative everywhere in that set. In a meanwhile, due to Kodzron [15], a function $f:D\to\mathbb{C}$ is called univalent on D if $f(z_1)\neq f(z_2)$ for all $z_1,z_2\in D$ with $z_1\neq z_2$.

For two analytic functions f and g in $\mathbb{D}$, the subordination between them is written as $f\prec g$. If f is subordinate to g, there is a Schwarz function ω with $\omega(0)=0$, $|\omega(z)|<1$, for all $z\in\mathbb{D}$, such that $f(z)=g(\omega(z))$. Note that, if g is univalent, then $f\prec g$ if and only if $f(0)=g(0)$ and $f(\mathbb{D})\subseteq g(\mathbb{D})$.

For each univalent function f, it can be represented by the Taylor series expansion as follows

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n,\quad |z|<1, \tag{1.1}$$

where $a_n\in\mathbb{C},\ \ n=2,3,\cdots$.

Even, according to Duren [6], every function $f\in S$ has an inverse that can be expressed as

$$f^{-1}(f(z))=z,\ \ z\in\mathbb{D}$$

and

$$f(f^{-1}(w))=w,\ \ |w|<r_0(f),r_0(f)\geq\frac{1}{4}$$

where

$$f^{-1}(w)=w-a_2w^2+(2a_2^2-a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\cdots\quad =g(w) \tag{1.2}$$

If both f and f^{-1} are univalent in $\mathbb{D}$, then the function $f\in A$ is called bi-univalent in $\mathbb{D}$. In this paper, we use σ as the class of bi-univalent functions in $\mathbb{D}$. The study of the class σ was initiated by a researcher named Lewin [16] around 1967 by estimated that $|a_2|<1.51$ and subsequently Brannan and Clunie [3] improved Lewin's result by proving that $|a_2|<\sqrt{2}$. Next, the study on the class σ was continued and studied in depth by such researchers [12, 15, 24].

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Received: 17 Nov. 2022
Accepted: 11 April 2023

10.11113/mjfas.v19n2.2807

By considering the function $f \in \sigma$, this study will focus on obtaining the results on the Hankel determinant. Noonan and Thomas [21] gave a definition for the mth Hankel determinant of f for integers $n \geq 1$ and $m \geq 1$ as

$$H_m(n) = \begin{vmatrix} a_n & \cdots & a_{n+m-1} \\ \vdots & \cdots & \vdots \\ a_{n+m-1} & \cdots & a_{n+2m-2} \end{vmatrix}, \quad a_1 = 1.$$

Taking into account the values of $m = 2$ and $n = 1$, Fekete and Szegö [7] represented the Hankel determinant of f as $H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix}$. Furthermore, if we take the values of $m = 2$, $n = 2$, we obtained the determinant of second Hankel determinant as follows

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = (a_2 a_4 - a_3^2).$$

In addition, by considering the values of $m = 2$, $n = 3$, we obtained the determinant $H_2(3)$.

There are many findings related to the results of $H_2(1)$, $H_2(2)$ and $H_2(3)$ for subclasses of univalent and bi-univalent functions have been widely explored by mathematicians, among them are as [5, 8, 10, 13, 17, 18, 20, 23].

Currently, many mathematicians work in the field of ordinary classical calculus, especially in the topic of quantum calculus or q-calculus. The earliest researcher studied q-calculus was Jackson [11]. He was also one of the earliest researchers developed the q-integral and q-derivative more systematically.

For a function $f \in A$ given by (1.1) and $0 < q < 1$, the q-derivative of a function f is defined by [1, 2] as follows

$$D_q\big(f(z)\big) = \frac{f(qz) - f(z)}{(q-1)z}, \quad z \neq 0,$$

$D_q\big(f(0)\big) = f'(0)$.

By taking an example $f(z) = z^k$ for k is a positive integer, the q-derivative of a function f is given by

$$D_q z^k = \frac{z^k - (zq)^k}{(1-q)z} = [k]_q z^{k-1}$$

As $q \to 1^-$ and $k \in \mathbb{N}$, we have

$$[k]_q = \frac{1-q^k}{1-q} = 1 + q + \cdots + q^{k-1} \to k.$$

Now, using the q-derivative of a function $f \in A$ and the principle of subordination, we introduce the subclass of A. We begin with the following definition first.

Definition 1.1 ([19]) Let φ be an analytic function with positive real part in the unit disk $\mathbb{D}$, satisfying $\varphi(0) = 1, \varphi'(0) > 0$ and symmetric with respect to the real axis. Such a function has series expansion of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \ (B_1 > 0) \tag{1.3}$$

Definition 1.2 A function $\mathrm{f} \in \sigma$ given by (1.1) is said to be in class q-$\mathcal{M}_\sigma(\alpha; \varphi)$, $0 \leq \alpha \leq 1$, φ given by (1.3) if

$$(1-\alpha)\left(\frac{zD_q\big(f(z)\big)}{f(z)}\right) + \alpha\left(1 + \frac{zqD_q\Big(D_q\big(f(z)\big)\Big)}{D_q f(z)}\right) \prec \varphi(z), \ \ z \in \mathbb{D} \tag{1.4}$$

and

$$(1-\alpha)\left(\frac{wD_q\big(g(w)\big)}{g(w)}\right) + \alpha\left(1 + \frac{wqD_q\Big(D_q\big(g(w)\big)\Big)}{D_q g(w)}\right) \prec \varphi(w), \quad w \in \mathbb{D} \tag{1.5}$$

where the function g is given by (1.2).

Next, we state some lemmas that are needed before we prove the main result in particular to get the upper bound for the second Hankel determinant for functions $\mathrm{f} \in$ q-$\mathcal{M}_\sigma(\alpha; \varphi)$.

Lemma 1.1 ([6]) Let $u(z)$ and $v(z)$ be analytic in the unit disk $\mathbb{D}$ with $u(0) = v(0) = 0$, $|u(z)| < 1$,

$|v(z) < 1|$, and suppose that

$$u(z) = \sum_{n=1}^{\infty} c_n z^n \quad and \quad v(z) = \sum_{n=1}^{\infty} d_n z^n \qquad (z \in \mathbb{D}) \tag{1.6}$$

Then $|c_n| \le 1$ and $|d_n| \le 1$ for all $n = 1, 2, 3, \cdots$.

Lemma 1.2 ([22]) If $p \in \mathcal{P}$ then $|p_k| \le 2$ for each $k \in \mathbb{N}$, where $\mathcal{P}$ is the family of all functions p analytic in $\mathbb{D}$ for which $Re\big(p(z)\big) > 0$,

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots,$$

for $z \in \mathbb{D}$.

Lemma 1.3 ([9]) If the function $p \in \mathcal{P}$, then

$$2p_2 = p_1^2 + x(4 - p_1^2)$$
$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1 x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)s,$$

for some x, s with $|x| \le 1$ and $|s| \le 1$.

Lemma 1.4 ([14]) Let $\psi(z) = \sum_{n=1}^{\infty} \psi_n z^n \in A$ be a Schwarz function so that $|\psi(z)| < 1$ for $|z| < 1$. Then

$$\psi_2 = x(1 - \psi_1^2)$$
$$\psi_3 = (1 - \psi_1^2)(1 - |x|^2)s - \psi_1(1 - \psi_1^2)x^2,$$

for some x, s with $|x| \le 1$ and $|s| \le 1$.

Main Results

Next, we state our main result. Before that, we get the values for the initial coefficients a_2, a_3 and a_4.

Lemma 2.1 For $0 \le \alpha \le 1$, let the function $f \in q\text{-}\mathcal{M}_\sigma(\alpha; \varphi)$ be given by (1.1). Then

$$a_2 = \frac{B_1 c_1}{(1 + \alpha)}$$

$$a_3 = \left[1 + \frac{1}{2}(q^2 - 1)\alpha\right]\left(\frac{B_1^2 c_1^2}{(1 + \alpha)^2}\right) + \frac{B_1(c_2 - d_2)}{[2q(q + 1) + q(q + 1)^3\alpha]}$$

and

$$\begin{aligned} a_4 = {} & \frac{[q(q + 1)(5q^3 - 1) + q^2(5q^6 + 10q^5 + 10q^4 + 4q^3 + q^2 + q + 1)\alpha]B_1^3 c_1^3}{[4q(q^2 + q + 1) + 2q(q + 1)(q^4 + 2q^3 + 3q^2 + 2q + 1)\alpha](1 + \alpha)^3} \\ & + \frac{5B_1^2 c_1(c_2 - d_2)}{(2q(q + 1)^2 + q(q + 1)^3\alpha)(1 + 2\alpha)} \\ & + \frac{B_1(c_3 - d_3)}{[2q(q^2 + q + 1) + q(q + 1)(q^4 + 2q^3 + 3q^2 + 2q + 1)\alpha]} \\ & + \frac{2B_2 c_1(c_2 + d_2)}{[2q(q^2 + q + 1) + q(q + 1)(q^4 + 2q^3 + 3q^2 + 2q + 1)\alpha]} \\ & + \frac{2B_3(c_1^3)}{[2q(q^2 + q + 1) + q(q + 1)(q^4 + 2q^3 + 3q^2 + 2q + 1)\alpha]} \end{aligned}$$

Proof The proof of this lemma begins by letting $f \in q\text{-}\mathcal{M}_\sigma(\alpha; \varphi)$ and $g = f^{-1}$. Next by using Lemma 1.1, there are two Schwarz functions $u, v : \mathbb{D} \to \mathbb{D}$ with

$$u(0) = v(0) = 0$$

which are given by (1.6). From Definition 1.2, (1.4) and (1.5), proof is continued by obtaining

$$(1 - \alpha)\left(\frac{zD_q\big(f(z)\big)}{f(z)}\right) + \alpha\left(1 + \frac{zqD_q\Big(D_q\big(f(z)\big)\Big)}{D_q f(z)}\right) = \varphi\big(u(z)\big) \tag{2.1}$$

and

$$(1 - \alpha)\left(\frac{wD_q\big(g(w)\big)}{g(w)}\right) + \alpha\left(1 + \frac{wqD_q\Big(D_q\big(g(w)\big)\Big)}{D_q g(w)}\right) = \varphi\big(v(z)\big) \tag{2.2}$$

We also have

$$\varphi\big(u(z)\big) = 1 + B_1 c_1 z + (B_1 c_2 + B_2 c_1^2)z^2 + (B_1 c_3 + 2c_1 c_2 B_2 + B_3 c_1^3)z^3 + \cdots \tag{2.3}$$
$$\varphi\big(v(z)\big) = 1 + B_1 d_1 w + (B_1 d_2 + B_2 d_1^2)w^2 + (B_1 d_3 + 2d_1 d_2 B_2 + B_3 d_1^3)w^3 + \cdots \tag{2.4}$$

By doing calculations to (2.1), (2.3) and (2.2), (2.4), it follows that

$$(1+q\alpha)qa_2 = B_1c_1 \tag{2.5}$$

$$(q+1)\big(q+\big(q^2(q+1)\big)\alpha\big)a_3 - q(1-(1-(q+1)^2)\alpha)a_2^2 = B_1c_2 + B_2c_1^2 \tag{2.6}$$

$$\begin{aligned}(q^3+q^2+q)(1+(q^3+q^2+q)\alpha)a_4 &- (q^2+2q)\big(1-\big(1-(q^2+q+1)(q+1)\big)\alpha\big)a_2a_3 \\ &+ q(1-(1-(q+1)^3)\alpha)a_2^3 \\ &= B_1c_3 + 2c_1c_2B_2 + B_3c_1^3\end{aligned} \tag{2.7}$$

and

$$-(1+q\alpha)qa_2 = B_1d_1 \tag{2.8}$$

$$\Big(q\big((2q+1)+(2q^3+3q^2)\alpha\big)\Big)a_2^2 - \big(q(q+1)(1+(q+1)\alpha)\big)a_3 = B_1d_2 + B_2d_1^2 \tag{2.9}$$

$$\begin{aligned}q\big((5q^2+4q+3) &+ (q(5q^4+9q^3+11q^2+4q+1)\alpha)\big)a_2a_3 \\ &- q\big((5q^2+3q+2)+(q^2(5q^3+8q^2+8q+1)\alpha)\big)a_2^3 \\ &- q(q(q^2+q+1)+q(q^4+2q^3+3q^2+2q+1)\alpha)a_4 \\ &= B_1d_3 + 2d_1d_2B_2 + B_3d_1^3\end{aligned} \tag{2.10}$$

Next, by performing a calculation on (2.5) and (2.8), we get that

$$c_1 = -d_1 \tag{2.11}$$

and

$$a_2 = \frac{B_1c_1}{(1+\alpha)} \tag{2.12}$$

Now, from (2.6) and (2.9), it follows that

$$a_3 = \left[1+\frac{1}{2}(q^2-1)\alpha\right]\left(\frac{B_1^2c_1^2}{(1+\alpha)^2}\right) + \frac{B_1(c_2-d_2)}{[2q(q+1)+q(q+1)^3\alpha]} \tag{2.13}$$

Also, from (2.7) and (2.10), we find that

$$\begin{aligned}a_4 &= \frac{[q(q+1)(5q^3-1)+q^2(5q^6+10q^5+10q^4+4q^3+q^2+q+1)\alpha]B_1^3c_1^3}{[4q(q^2+q+1)+2q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)^3} \\ &+ \frac{5B_1^2c_1(c_2-d_2)}{(2q(q+1)^2+q(q+1)^3\alpha)(1+2\alpha)} + \frac{B_1(c_3-d_3)}{[2q(q^2+q+1)+q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha]} \\ &+ \frac{2B_2c_1(c_2+d_2)}{[2q(q^2+q+1)+q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha]} \\ &+ \frac{2B_3(c_1^3)}{[2q(q^2+q+1)+q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha]}\end{aligned} \tag{2.14}$$

The proof of Lemma 2.1 is completed. □

The main result is stated as follows.

Theorem 2.1 For $0 \le \alpha \le 1$, let the function $f \in q\text{-}\mathcal{M}_\sigma(\alpha;\varphi)$ be given by (1.1). Then

$$|a_2a_4 - a_3^2| \le B_1\begin{cases}\dfrac{4B_1}{(2q(q+1)+q(q+1)^3\alpha)^2} & Q\le 0, P\le -Q \\ \left|\dfrac{-4B_1^3}{[4q(q^2+q+1)+2q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)^3} + \dfrac{2B_3}{[2q(q^2+q+1)+q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)}\right| & \left(Q\ge 0, P\ge -\dfrac{Q}{2}\right), or\ (Q\le 0, P\ge -Q) \\ \dfrac{4PR-Q^2}{4P} & Q>0, P\le -\dfrac{Q}{2}\end{cases}$$

where

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$$P = \left|\frac{-4B_1^3}{[4q(q^2+q+1)+2q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)^3} + \frac{2B_3}{[2q(q^2+q+1)+q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)}\right| - 2\left(\frac{8B_1^2}{(2q(q+1)^2+q(q+1)^3\alpha)^2(1+2\alpha)} + \frac{|2B_2|}{[2q(q^2+q+1)+q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)}\right) + \left(\frac{-2B_1}{(2q(q^2+q+1)+q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha)(1+\alpha)}\right) + \left(\frac{4B_1}{(2q(q+1)+q(q+1)^3\alpha)^2}\right)$$

$$Q = 2\left(\frac{8B_1^2}{(2q(q+1)^2+q(q+1)^3\alpha)^2(1+2\alpha)} + \frac{|2B_2|}{(2q(q^2+q+1)+q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha)(1+\alpha)}\right) + \left(\frac{2B_1}{(2q(q^2+q+1)+q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha)(1+\alpha)}\right) - 4\left(\frac{2B_1}{(2q(q+1)+q(q+1)^3\alpha)^2}\right)$$

$$R = \frac{4B_1}{(2q(q+1)+q(q+1)^3\alpha)^2}$$

Proof

From Lemma 2.1 or in other words, by considering equations (2.12), (2.13) and (2.14), we can establish that

$$|a_2a_4 - a_3^2| = \left|\left[\frac{-4B_1^4c_1^4}{[4q(q^2+q+1)+2q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)^3}\right] + \left[\frac{8B_1^3c_1^2(c_2-d_2)}{(2q(q+1)^2+q(q+1)^3\alpha)^2(1+2\alpha)}\right] + \frac{2B_1B_2c_1^2(c_2+d_2)}{[2q(q^2+q+1)+q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)} + \frac{2B_1B_3c_1^4}{[2q(q^2+q+1)+q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)} + \frac{B_1^2c_1(c_3-d_3)}{[2q(q^2+q+1)+q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)} - \frac{B_1^2(c_2-d_2)^2}{(2q(q+1)+q(q+1)^3\alpha)^2}\right| \quad (2.15)$$

Due to Lemma 1.4 and (2.11), we find

$$c_2 = x(1-c_1^2)$$
$$d_2 = y(1-d_1^2)$$

where

$$c_2 - d_2 = (1-c_1^2)(x-y) \quad (2.16)$$
$$c_2 + d_2 = (1-c_1^2)(x+y) \quad (2.17)$$

and

$$c_3 = (1-c_1^2)(1-|x|^2)s - c_1(1-c_1^2)x^2$$
$$d_3 = (1-d_1^2)(1-|y|^2)w - d_1(1-d_1^2)y^2$$

where

$$c_3 - d_3 = (1-c_1^2)[(1-|x|^2)s - (1-|y|^2)w] - c_1(1-c_1^2)(x^2+y^2) \quad (2.18)$$

for some x, y, s, w with $|x| < 1, |y| < 1, |s| < 1$ and $|w| < 1$.

Next, by substituting equation (2.16), (2.17) and (2.18) in (2.15), we get

$$|a_2a_4 - a_3^2| = B_1 \left|\left[\frac{-4B_1^3}{[4q(q^2+q+1)+2q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)^3} + \frac{2B_3}{[2q(q^2+q+1)+q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)}\right]c_1^4 + \left[\frac{8B_1^2(x-y)}{(2q(q+1)^2+q(q+1)^3\alpha)^2(1+2\alpha)} + \frac{2B_2(x+y)}{[2q(q^2+q+1)+q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)}\right] \times c_1^2(1-c_1^2) - \frac{B_1c_1^2(1-c_1^2)}{[2q(q^2+q+1)+q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)}(x^2+y^2) - \frac{B_1(1-c_1^2)^2}{(2q(q+1)+q(q+1)^3\alpha)^2(1+\alpha)}(x-y)^2 + \frac{B_1c_1(1-c_1^2)}{[2q(q^2+q+1)+q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)}[(1-|x|^2)s - (1-|y|^2)w]\right|$$

Since $|c_1| \le 1$, letting $c_1 = c$, we may assume without loss of generality that $c \in [0,1]$. Next, it can be expressed that

$$|a_2a_4 - a_3^2| \le B_1 \left(\left|\left[\frac{-4B_1^3}{[4q(q^2+q+1)+2q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)^3} + \frac{2B_3}{[2q(q^2+q+1)+q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)}\right]\right|c^4 + \left[\frac{8B_1^2}{(2q(q+1)^2+q(q+1)^3\alpha)^2(1+2\alpha)} + \frac{|2B_2|}{[2q(q^2+q+1)+q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)}\right]c^2(1-c^2)(|x| + |y|) + \frac{B_1c^2(1-c^2)}{[2q(q^2+q+1)+q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)}(|x|^2+|y|^2) + \frac{B_1(1-c^2)^2}{(2q(q+1)+q(q+1)^3\alpha)^2}(|x|+|y|)^2 + \frac{B_1c(1-c^2)}{[2q(q^2+q+1)+q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)}[(1-|x|^2)|s| + (1-|y|^2)|w|]\right)$$

$$|a_2a_4 - a_3^2| \le B_1 \left(\left|\left[\frac{-4B_1^3}{[4q(q^2+q+1)+2q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)^3} + \frac{2B_3}{[2q(q^2+q+1)+q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)}\right]\right|c^4 + \left[\frac{8B_1^2}{(2q(q+1)^2+q(q+1)^3\alpha)^2(1+2\alpha)} + \frac{|2B_2|}{[2q(q^2+q+1)+q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)}\right]c^2(1-c^2)(|x| + |y|) + \frac{B_1c^2(1-c^2)}{[2q(q^2+q+1)+q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)}(|x|^2+|y|^2) + \frac{B_1(1-c^2)^2}{(2q(q+1)+q(q+1)^3\alpha)^2}(|x|+|y|)^2 + \frac{B_1c(1-c^2)}{[2q(q^2+q+1)+q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)}[(1-|x|^2) + (1-|y|^2)]\right)$$

$$\le B_1\Bigg(\Bigg|\Bigg[\frac{-4B_1^3}{[4q(q^2+q+1)+2q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)^3} + \frac{2B_3}{[2q(q^2+q+1)+q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)}\Bigg]\Bigg|c^4 + \frac{2B_1c(1-c^2)}{[2q(q^2+q+1)+q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)} + \Bigg[\frac{8B_1^2}{(2q(q+1)^2+q(q+1)^3\alpha)^2(1+2\alpha)} + \frac{|2B_2|}{[2q(q^2+q+1)+q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)}\Bigg]c^2(1-c^2)(|x| + |y|) + \Bigg[\frac{B_1c^2(1-c^2)}{[2q(q^2+q+1)+q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)} - \frac{B_1c(1-c^2)}{[2q(q^2+q+1)+q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)}\Bigg](|x|^2+|y|^2) + \frac{B_1(1-c^2)^2}{(2q(q+1)+q(q+1)^3\alpha)^2}(|x|+|y|)^2\Bigg)$$

Now, by replacing $\lambda = |x| \le 1$ and $\mu = |y| \le 1$, we obtain

$$|a_2a_4 - a_3^2| \le B_1[T_1 + (\lambda+\mu)T_2 + (\lambda^2+\mu^2)T_3 + (\lambda+\mu)^2T_4] = B_1F(\lambda,\mu)$$

where

$$T_1 = T_1(c) = \Bigg|\Bigg[\frac{-4B_1^3}{[4q(q^2+q+1)+2q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)^3} + \frac{2B_3}{[2q(q^2+q+1)+q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)}\Bigg]\Bigg|c^4 + \frac{2B_1c(1-c^2)}{[2q(q^2+q+1)+q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)} \ge 0$$

$$T_2 = T_2(c) = \Bigg[\frac{8B_1^2}{(2q(q+1)^2+q(q+1)^3\alpha)^2(1+2\alpha)} + \frac{|2B_2|}{[2q(q^2+q+1)+q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)}\Bigg]c^2(1-c^2) \ge 0$$

$$T_3 = T_3(c) = \frac{B_1c(c-1)(1-c^2)}{[2q(q^2+q+1)+q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)} \le 0$$

$$T_4 = T_4(c) = \frac{B_1(1-c^2)^2}{(2q(q+1)+q(q+1)^3\alpha)^2} \ge 0$$

To get the upper bound of $|a_2a_4 - a_3^2|$, we need to maximize the function $F(\lambda,\mu)$ on the closed square $[0,1]\times[0,1]$ for $c \in [0,1]$. It means that the maximum of $F(\lambda,\mu)$ needs to be investigated according to $c \in (0,1), c = 0$ and $c = 1$ by taking into account the sign of $F_{\lambda\lambda}F_{\mu\mu} - (F_{\lambda\mu})^2$.

After doing the calculations, we obtain that

$$\begin{aligned} F_\lambda &= T_2 + 2\lambda T_3 + 2\lambda T_4 + 2\mu T_4 \\ F_\mu &= T_2 + 2\mu T_3 + 2\lambda T_4 + 2\mu T_4 \\ F_{\lambda\lambda} &= 2T_3 + 2T_4 \\ F_{\mu\mu} &= 2T_3 + 2T_4 \\ F_{\lambda\mu} &= 2T_4 \end{aligned}$$

Let $c \in [0,1)$. Since $T_3 + 2T_4 > 0$ for $\alpha \in [0,1]$ and $T_3 < 0$, we conclude that

$$\begin{aligned} F_{\lambda\lambda}F_{\mu\mu} - (F_{\lambda\mu})^2 &= (2T_3+2T_4)^2 - (2T_4)^2 \\ &= 4T_3^2 + 8T_3T_4 + 4T_4^2 - 4T_4^2 \\ &= 4T_3(T_3+2T_4) \\ &< 0 \end{aligned}$$

The conclusion is the function F cannot have a local maximum in the interior of the square.

Next, we go to the next step by investigating the maximum of $F(\lambda,\mu)$ on the boundary of the square. For $\lambda = 0$ and $0 \le \mu \le 1$ (similarly $\mu = 0$ and $0 \le \lambda \le 1$), we get

$$F(0,\mu) = H(\mu) = (T_3 + T_4)\mu^2 + T_2\mu + T_1.$$

i) If $T_3 + T_4 \ge 0$: It is clear that $H'(\mu) = 2(T_3 + T_4)\mu + T_2 > 0$ for $0 < \mu < 1$ and any fixed $c \in [0,1)$. In conclusion, $H(\mu)$ is an increasing function. Hence, for fixed $c \in [0,1)$, we conclude that the maximum of $H(\mu)$ occurs at $\mu = 1$, and

$$\max H(\mu) = H(1) = T_1 + T_2 + T_3 + T_4.$$

ii) If $T_3 + T_4 < 0$: Then we consider for critical point $\mu = \frac{-T_2}{2(T_3+T_4)} = \frac{T_2}{2\theta}$ for fixed $c \in [0,1)$ where $\theta = -(T_3 + T_4) > 0$.

In this case, two cases are considered separately.

Case 2.1 Let $\mu = \frac{T_2}{2\theta} > 1$. Then $\theta < \frac{T_2}{2} \le T_2$, and so $T_2 + T_3 + T_4 \ge 0$. Therefore,

$$H(0) = T_1 \le T_1 + T_2 + T_3 + T_4 = H(1).$$

Case 2.2 Let $\mu = \frac{T_2}{2\theta} \le 1$. Since $\frac{T_2}{2} \ge 0$, we get $\frac{T_2^2}{4\theta} \le \frac{T_2}{2} \le T_2$. Also we have $H(1) = T_1 + T_2 + T_3 + T_4 \le T_1 + T_2$, Therefore,

$$H(0) = T_1 \le T_1 + \frac{T_2^2}{4\theta} = H\left(\frac{T_2}{2}\right) \le T_1 + T_2.$$

In a meanwhile, for $c = 1$, we obtain

$$F(\lambda,\mu) = \left| \frac{-4B_1^3}{[4q(q^2+q+1) + 2q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)^3} + \frac{2B_3}{[2q(q^2+q+1) + q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)} \right|.$$

By considering above equation and cases (i) and (ii), for $0 \le \mu \le 1$ for fixed $c \in [0,1]$, $H(\mu)$ attains its maximum when $T_3 + T_4 \ge 0$, it means

$$\max H(\mu) = H(1) = \underbrace{(T_3 + T_4)}_{\ge 0} + T_2 + T_1.$$

For $\lambda = 1$ and $0 \le \mu \le 1$ (similarly $\mu = 0$ and $0 \le \lambda \le 1$), we get

$$F(1,\mu) = G(\mu) = (T_3 + T_4)\mu^2 + (T_2 + 2T_4)\mu + T_1 + T_2 + T_3 + T_4.$$

iii) If $T_3 + T_4 \ge 0$: It is clear that $G'(\mu) = 2(T_3 + T_4)\mu + T_2 + 2T_4 > 0$ for $0 < \mu < 1$ and any fixed $c \in [0,1)$. In conclusion, $G(\mu)$ is an increasing function. Hence, for fixed $c \in [0,1)$, the maximum of $G(\mu)$ occurs at $\mu = 1$, and

$$\max G(\mu) = G(1) = T_1 + 2T_2 + 2T_3 + 4T_4.$$

iv) If $T_3 + T_4 < 0$: Then we consider for critical point $\mu = \frac{-(T_2+2T_4)}{2(T_3+T_4)} = \frac{(T_2+2T_4)}{2\theta}$ for fixed $c \in [0,1)$ where $\theta = -(T_3 + T_4) > 0$.

Again, two cases are considered separately.

Case 2.3 Let $\mu = \frac{(T_2+2T_4)}{2\theta} > 1$. Then $\theta < \frac{T_2+2T_4}{2} \le T_2 + 2T_4$, and so $T_2 + T_3 + 3T_4 \ge 0$. Therefore,

$$G(0) = T_1 + T_2 + T_3 + T_4 \le T_1 + T_2 + T_3 + T_4 + T_2 + T_3 + 3T_4 = G(1).$$

Case 2.4 Let $\mu = \frac{(T_2+2T_4)}{2\theta} \le 1$. Since $\frac{(T_2+2T_4)}{2} \ge 0$, we get that

$$\frac{(T_2 + 2T_4)^2}{4\theta} \le \frac{T_2 + 2T_4}{2} \le T_2 + 2T_4.$$

Therefore,

$$\begin{aligned} G(0) &= T_1 + T_2 + T_3 + T_4 \le T_1 + T_2 + T_3 + T_4 + \frac{(T_2 + 2T_4)^2}{4\theta} \\ &= G\left(\frac{T_2 + 2T_4}{2\theta}\right) \le T_1 + T_2 + T_3 + T_4 + T_2 + 2T_4 = T_1 + 2T_2 + T_3 + 3T_4 \\ &= T_1 + 2T_2 + \underbrace{(T_3 + T_4)}_{<0} + 2T_4 \end{aligned}$$

By considering cases (iii) and (iv), for $0 \le \mu \le 1$ for fixed $c \in [0,1]$, $G(\mu)$ gets its maximum when $T_3 + T_4 \ge 0$, it means

$$\max G(\mu) = G(1) = T_1 + 2T_2 + 2\underbrace{(T_3 + T_4)}_{\ge 0} + 2T_4.$$

Since $H(1) \le G(1)$ for $c \in [0,1]$, max $F(\lambda,\mu) = F(1,1)$ on the boundary of the square. Thus, the

maximum of F occurs at $\lambda = 1$ and $\mu = 1$ in the closed square.
Let $K: [0,1] \to \mathbb{R}$ be given by

$$K(c) = B_1 \max F(\lambda, \mu) = B_1 F(1,1) = B_1(T_1 + 2T_2 + 2T_3 + 4T_4). \tag{2.19}$$

Substituting the values of T_1, T_2, T_3 and T_4 into the function $K(c)$ defined by (2.19) yields

$$\begin{aligned} K(c) = B_1 \Bigg\{ \Bigg[\Bigg| & \frac{-4B_1^3}{[4q(q^2+q+1) + 2q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)^3} \\ & + \frac{2B_3}{[2q(q^2+q+1) + q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)} \Bigg| \\ & - 2\left(\frac{8B_1^2}{(2q(q+1)^2 + q(q+1)^3\alpha)^2(1+2\alpha)} \right. \\ & \left. + \frac{|2B_2|}{[2q(q^2+q+1) + q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)} \right) \\ & + \left(\frac{-2B_1}{(2q(q^2+q+1) + q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha)(1+\alpha)} \right) \\ & + \left(\frac{4B_1}{(2q(q+1) + q(q+1)^3\alpha)^2} \right) \Bigg] c^4 \\ & + \Bigg[2\left(\frac{8B_1^2}{(2q(q+1)^2 + q(q+1)^3\alpha)^2(1+2\alpha)} \right. \\ & \left. + \frac{|2B_2|}{(2q(q^2+q+1) + q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha)(1+\alpha)} \right) \\ & + \left(\frac{2B_1}{(2q(q^2+q+1) + q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha)(1+\alpha)} \right) \\ & - 4\left(\frac{2B_1}{(2q(q+1) + q(q+1)^3\alpha)^2} \right) \Bigg] c^2 + \frac{4B_1}{(2q(q+1) + q(q+1)^3\alpha)^2} \Bigg\} \end{aligned}$$

Now, write $c^2 = t$ and

$$\begin{aligned} P = & \left| \frac{-4B_1^3}{[4q(q^2+q+1) + 2q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)^3} \right. \\ & \left. + \frac{2B_3}{[2q(q^2+q+1) + q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)} \right| \\ & - 2\left(\frac{8B_1^2}{(2q(q+1)^2 + q(q+1)^3\alpha)^2(1+2\alpha)} \right. \\ & \left. + \frac{|2B_2|}{[2q(q^2+q+1) + q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha](1+\alpha)} \right) \\ & + \left(\frac{-2B_1}{(2q(q^2+q+1) + q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha)(1+\alpha)} \right) \\ & + \left(\frac{4B_1}{(2q(q+1) + q(q+1)^3\alpha)^2} \right) \end{aligned} \tag{2.20}$$

$$\begin{aligned} Q = & \; 2\left(\frac{8B_1^2}{(2q(q+1)^2 + q(q+1)^3\alpha)^2(1+2\alpha)} \right. \\ & \left. + \frac{|2B_2|}{(2q(q^2+q+1) + q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha)(1+\alpha)} \right) \\ & + \left(\frac{2B_1}{(2q(q^2+q+1) + q(q+1)(q^4+2q^3+3q^2+2q+1)\alpha)(1+\alpha)} \right) \\ & - 4\left(\frac{2B_1}{(2q(q+1) + q(q+1)^3\alpha)^2} \right) \end{aligned} \tag{2.21}$$

$$R = \frac{4B_1}{(2q(q+1) + q(q+1)^3\alpha)^2} \tag{2.22}$$

By considering

$$\max_{0\le t\le 1}\left(Pt^2 + Qt + R\right) = \begin{cases} R & Q \le 0, P \le -Q \\ P+Q+R & \left(Q \ge 0. P \ge -\frac{Q}{2}\right), or, (Q \le 0, P \ge -Q) \\ \frac{4PR-Q^2}{4P} & Q > 0, P \le -\frac{Q}{2} \end{cases}$$

Finally, we can concude that

$$|a_2a_4 - a_3^2| \le B_1 \begin{cases} R & Q \le 0, P \le -Q \\ P+Q+R & \left(Q \ge 0. P \ge -\frac{Q}{2}\right), or, (Q \le 0, P \ge -Q) \\ \frac{4PR-Q^2}{4P} & Q > 0, P \le -\frac{Q}{2} \end{cases}$$

where P, Q and R are given by (2.20), (2.21) and (2.22).
The proof of Theorem 2.1 is completed. □

By putting $q = 1$ in Theorem 2.1, we obtain the following corollary.

Corollary 2.1 ([14]) For $0 \le \alpha \le 1$, let the function $f \in \mathcal{M}_\sigma(\alpha;\varphi)$ be given by (1.1). Then

$$|a_2a_4 - a_3^2| \le B_1 \begin{cases} \frac{B_1}{(2+4\alpha)^2} & Q \le 0, P \le -Q \\ \left|\frac{-B_1^3}{(3+9\alpha)(1+\alpha)^3} + \frac{B_3}{(3+9\alpha)(1+\alpha)}\right| & \left(Q \ge 0, P \ge -\frac{Q}{2}\right), or\ (Q \le 0, P \ge -Q) \\ \frac{4PR-Q^2}{4P} & Q > 0, P \le -\frac{Q}{2} \end{cases}$$

where

$$P = \left|\frac{-B_1^3}{[3+9\alpha](1+\alpha)^3} + \frac{B_3}{[3+9\alpha](1+\alpha)}\right| - 2\left(\frac{B_1^2}{8(1+\alpha)^2(1+2\alpha)} + \frac{|B_2|}{[3+9\alpha](1+\alpha)}\right) + \left(\frac{-2B_1}{2(3+9\alpha)(1+\alpha)}\right) + \left(\frac{B_1}{(2+4\alpha)^2}\right)$$

$$Q = 2\left(\frac{B_1^2}{8(1+\alpha)^2(1+2\alpha)} + \frac{|B_2|}{(3+9\alpha)(1+\alpha)}\right) + \left(\frac{2B_1}{2(3+9\alpha)(1+\alpha)}\right) - \left(\frac{2B_1}{2(2+4\alpha)^2}\right)$$

$$R = \frac{B_1}{(2+4\alpha)^2}$$

Conclusions

In conclusion, we obtained the initial coefficients a_2, a_3 and a_4 and the upper bound for the functional $|a_2a_4 - a_3^2|$ of functions $f \in q\text{-}\mathcal{M}_\sigma(\alpha;\varphi)$ be given by (1.1).

Conflicts of Interest

The author(s) declare(s) that there is no conflict of interest regarding the publication of this paper.

Acknowledgment

The work presented here was partially supported by SBK0485-2021.

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