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RESEARCH ARTICLE

Simulating Rubber Prices under Geometric Fractional Brownian Motion with Different Hurst Estimators

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Abstract This research focuses on the simulation of rubber prices using geometric Brownian motion (GBM) and geometric fractional Brownian motion (GFBM). An analysis of the Hurst values estimated using three different methods, namely, rescaled range analysis, aggravate variance, and residuals of regression. These methods are based on the slope deviation technique. An error estimation via Mean Absolute Percentage Error (MAPE) is computed to analyse the accuracy of the simulation models which shows that the GBM is highly accurate over a 120-day period, and as the number of days increases, the accuracy decreases. Moreover, the rescaled range analysis method can estimate the Hurst values for all five ranges of days in comparison to the residuals of regression and the aggregate variance methods. The rescaled range analysis method also able to produce a highly accurate simulation up to 150-day

Keywords: Rubber Prices, Geometric Brownian Motion, Geometric Fractional Brownian Motion, Hurst Exponent.

Introduction

According to the Cambridge Business English Dictionary, commodity is defined as a product that can be traded in large quantities such as oil, metal, coffee, etc under the stock market division. Similarly in finance, commodity is defined as a financial product that can be traded. In Malaysia, the exports could be divided to 3 sections, the manufactured products, agriculture products and the mining products. The sections and sub-sections are as shown in Figure 1. According to the Malaysian External Trade Statistic Bulletin based on June 2021 [1], the principal goods in exports include Electrical & Electronic (E&E) products, palm oil and palm oil-based products, refined petroleum products, crude petroleum, Liquified Natural Gas (LGN), wood and timber-based products, and natural rubber. Agriculture exports, which accounted for 8.3% of total exports, surged by 40.0% from RM6.3 billion in June 2020 to RM8.8 billion.

Natural rubber was a vital pillar of Malaysia's export-oriented economy throughout much of the twentieth century, according to the Economic History of Malaya (EHM) website, the worldwide demand for natural rubber is expected to expand at a CAGR of 4.8% in the future (2019–2023). Therefore, it could be concluded that rubber is one of the commodities that should be monitored. Thus, the forecasting of the rubber price should be analysed for future use. Natural rubber has the third greatest consistency among main export items, and in Malaysia there are grades SMR20, SMR CV, SMR L, SMR 5, SMR GP and SMR 10. In this study, we used historical prices of grade SMR20

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rubber to simulate the prices over 5 different periods, namely 60, 90, 120, 180, and 180 days, using geometric Brownian motion (GBM) and geometric fractional Brownian motion (GFBM) models, where for the latter, the Hurst parameters are first estimated using the rescaled range analysis, residuals of regression, and aggregate variance.

Reference [2] focused on the mechanisms involved in the anticipating stock prices. The algorithm involved in prediction utilising the Geometric Brownian Motion model was presented in depth. Aside from that, the use of log-normal distributions in modelling too were discussed which were used as an inspiration for this research analysis. Next, [3] studied the Hurst Exponent Estimators' Method algorithm such as the rescaled range, aggregated variance, absolute moment method that uses deviation from slope techniques, methods that uses filtration techniques such as discrete variance, variance versus level and second order discrete derivative using wavelets as well as other methods such as Higuchi, periodogram, and finally the variance of regression residuals method. Moreover, [4] investigated the accuracy of GBM and GFBM in modelling Malaysia's crude palm oil simulation. An error analysis was also performed in order to study the accuracy and conclude the best model in terms of commodity price simulation by calculating the mean-absolute percentage error (MAPE) value for each model as well as each Hurst exponents that were calculated using different methods which enables the learning of the GFBM method in prediction of commodity prices.

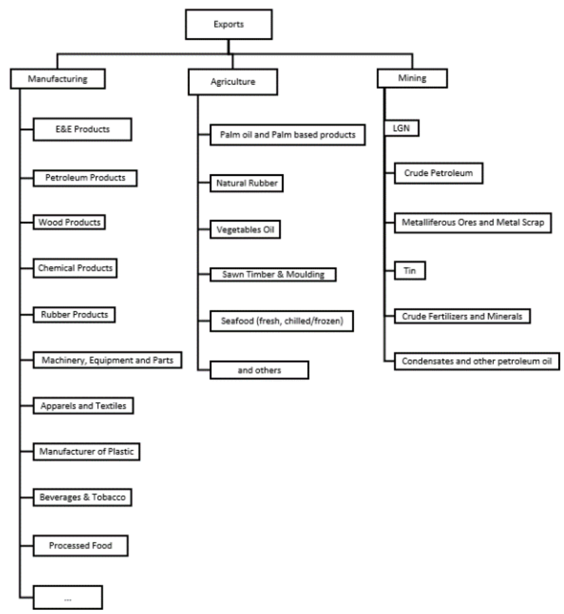


Figure 1. List of Sections and Subsections of Exports [1]

Geometric Brownian Motion

Geometric Brownian motion (GBM) is a continuous time stochastic model under the Brownian motion which are commonly known as the Wiener Process. Brownian motion describes the random movements of particles from the collision with the surrounding molecules. Similarly, when it comes to finance, the random movement in behaviours with respect to time is observed such as the random fluctuation on the stocks. In a nutshell, GBM is a model that is popularly used in predicting the future prices of a stock in finance. The GBM in stochastic differential equation is

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dW_t \qquad (1)$$

Derivation of the GBM analytical solution:
Given $\sigma_t = \sigma$ and $\mu_t = \mu$.
Thus, (1) can be written as

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \qquad (2)$$

According to the basic differentiation rules, we could note that

$$\frac{d}{dS_t}(lnS_t) = \frac{1}{S_t}$$

Rearranging it, we have

$$d(lnS_t) = \frac{dS_t}{S_t} \quad (3)$$

Substituting (3) into (1),

$$dln(S_t) = \mu dt + \sigma dW_t \quad (4)$$

From (2),

$$dS_t = (\mu dt + \sigma dW_t)S_t \quad (5)$$

So,

$$(dS_t)^2 = S_t^2[(\mu dt)^2 + (\sigma dW_t)^2 + 2\sigma\mu dtdW_t] \quad (6)$$

Imposing conditions $(dS_t)^2 = 0$, $(dW_t)^2 = dt$ and $dtdW_t = 0$ into (6),

$$(dS_t)^2 = \sigma^2 S_t^2 \quad dt \quad (7)$$

$$df = \frac{\partial f}{\partial t}dt + f'dS_t + \frac{1}{2}f''(dS_t)^2 \quad (8)$$

When $f = \ln(S_t)$, Equation (8) becomes

$$d\ln(S_t) = \frac{\partial \ln(S_t)}{\partial t}dt + \frac{1}{S_t}dS_t + \frac{1}{2}\left(-\frac{1}{S_t^2}\right)(dS_t)^2 \quad (9)$$

Substituting (5) and (7) into (9) yields

$$d\ln(S_t) = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t \quad (10)$$

Using $dln(S_t) = \ln(S_t) - \ln(S_0)$,

Integration with t=0 as starting and t=t as end point

$$\ln\left(\frac{S_t}{S_0}\right) = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW_t$$

Since LHS is an integrated solution, the RHS also should be integrated

$$\ln\left(\frac{S_t}{S_0}\right) = \left(\mu - \frac{1}{2}\sigma^2\right)t + W_t$$

Taking S_t as the subject, the analytical solution is

$$S_t = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t} \quad (11)$$

where at $t = 0$, $S_0 =$ initial stock price (constant), $W_t =$ Standard Brownian Motion, $\mu =$ constant drift and $\sigma =$ constant volatility

Geometric Fractional Brownian Motion

Geometric fractional Brownian motion (GFBM) model is an extension of the Brownian motion as well as the generalization of GBM model. While modelling using GBM, 1-dimensional distributions often are log normal and the log returns are independent random normal variables. According to [2], log returns normally contains long-range dependency property. Thus, the solution to this would be the replacement of the standard Brownian motion with the fractional Brownian motion, B_H. GFBM is a continuous time Gaussian Process. Gaussian process and the model could be presented in stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dB_t^H \quad (7)$$

with covariance $E[B_H(t)B_H(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$ and the solution

$$S_t = S_0 e^{\left(\sigma B^H(t) + \mu t - \frac{1}{2}\sigma^2 t^{2H}\right)}$$

where H is a real number in (0,1)

Hurst Exponent

Hurst exponent or also known as the Hurst parameter, H, in this paper could be observed being mentioned in the model of GFBM both stochastic differential equation as well as in the analytic solution. The research with Hurst exponent began with Harold Edwin Hurst's endeavour to develop a way to manage the natural flow of water after spending 60 years studying the Nile River for the Egyptian government [3]. An attempt is made to establish a reservoir to manage the natural flow of water, which is one of nature's unpredictability. Because a reservoir is a body of water generated by a dam, an ideal dam was determined to be built to achieve optimal dam size that would ensure that the outflow is uniform, and the dam never overflows [4].

Aside from that, Hurst exponent is also used in fractal geometry where H or H_q is known as the generalized hurst exponent. According to [5], Hurst exponent can be applied in the calculation of fractal dimensions, $D = 2 - H$, where *D* is the fractal dimension and *H* is the slope of the line that may be calculated using R/S analysis method, Moreover, calculated using Hurst exponents are beneficial in defining the degree of unpredictability of yield time series. For example, [6] investigated the Covid-19 spread using fractal interpolation.

Hurst exponent measures the long-term memory of a time series, that is, the degree by which that series deviates from a random walk. H is a real number that has been examined using several estimator methods in the range (0,1) where when:

$H = 0.5$: The process is a Brownian Motion/Wiener Process with no correlation
$H < 0.5$: The process is long-range dependent with negative correlation or anti-persistent
$H > 0.5$: The process is short-range dependent with positive correlation or persistent

Methodology

Data

The modelling methods are centred on commodities pricing. After doing extensive study, rubber of grade SMR20 was selected, and the daily price history was obtained from the Malaysian Rubber Board official website [7]. The historical prices are chosen from January until August 2021, which are then divided into 60, 90-, 120-, 150- and 180-day segments. These ranges are bigger than the ranges used in other studies [11].

Geometric Brownian Motion Simulation

The steps in simulating the rubber prices by Geometric Brownian Motion (GBM) using MATLAB are:

1. The logarithmic return of the rubber price, P_t, is calculated using $R_t = ln\left(\frac{P_t}{P_{t-1}}\right)$, for $t = 1,2,\dots,n$
2. The volatility and drift are calculated with drift, $\mu = \frac{1}{n}(\sum_1^n R_t)$ and volatility, $\sigma = \sqrt{\frac{\sum(R_t-\mu)^2}{N}}$
3. The Brownian motion, Z_t, the normal random number with mean = 0 and standard deviation = 1
4. The Price is finally simulated using $S_t = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma Z_t}$ where T is count of the data used, μ is drift, σ is volatility and S_0 is the initial rubber price

Geometric Fractional Brownian Motion Simulation

The simulation of rubber prices is done through MATLAB according to the following steps:

1. Calculate the sample drift, $\hat{\mu}$, and sample volatility, $\hat{\sigma}$, using the formulas $\hat{\sigma} = \frac{\sigma}{\sqrt{|\Delta t|^{2H}}}$ and $\hat{\mu} = \frac{\mu}{\Delta t} + \frac{\hat{\sigma}^2}{2}$ where the σ is the volatility calculated in GBM, μ is drift calculated in GFBM, H is Hurst exponent and $\Delta t = t_2 - t_1 = \cdots = t_n - t_{n-1}$
2. Generate pseudo random number, 2 x 2 Sobol Sequence.
3. Obtain the Random Complex Number
4. Calculate the fractal Gaussian noise (fGn)
5. Calculate the cumulative sum of it, W_j
6. Finally, simulate the price using the formula,

$$S_t = S_0 \exp\left(r\frac{jT}{n} - \frac{1}{2}\sigma^2\left(\frac{jT}{n}\right)^{2H} + \sigma\left(\frac{T}{n}\right)^{H} W_j\right)$$

Hurst Exponent Estimation Methods

The methods used in this paper in estimating the Hurst Exponent are the Rescaled Range, Aggregate Variance and the Residuals of Regression method which has the similarities of using the deviation slope to estimate. The difference however is based on the parameters calculated as in the variables used in the graphs that plotted in log-log scale. The hurst exponent in the rescaled range is the slope of the log (R/S) and log *n* whereas for aggregate variance method the hurst exponent is the solution of $\frac{slope}{2} + 1$ from the hurst exponent is log of Variance vs log of *m* on log-log scale. For residuals of regression method, the hurst exponent is half of the slope of the graph log of Residual Variance vs log of size of blocks on log-log scale. The slope for all the 3 methods is based on fitting the straight line on the graphs.

The Hurst Exponents are calculated using 3 different methods, the Rescaled Range, Aggregate Variance, and the Residuals of Regression Method based on [8] and [9].

Rescaled Range Analysis Method

1. Define the price data as $X = X_1, X_2, \ldots, X_n$
2. Calculate the mean, *m*, $m = \frac{1}{n}\sum_{i=1}^{n} X_i$ where X_i is the rubber price for $t = 1,2, \ldots, n$
3. Create a mean adjusted series $Y_t = X_t - m$ for $t = 1,2, \ldots, n$
4. Calculate cumulative deviate series $Z_t = \sum_{i=1}^{n} Y_i$ for $t = 1,2, \ldots, n$
5. Create range series, R using $R_t = \max(Z_{1,}Z_{2,\ldots,}Z_t) - \min(Z_{1,}Z_{2,\ldots,}Z_t)$
6. Create Standard deviation series, S, using $S_t = \sqrt{\frac{1}{t}\sum_{i=1}^{t}(X_i - m(t))^2}$ for $t = 1,2, \ldots, n$
7. Calculate rescaled range series $(R/S)_t = \frac{R_t}{S_t}$ for $t = 1,2, \ldots, n$
8. Plot a graph for log of $(R/S)_t$ vs log of Size of the Block and fit a straight line.
9. Obtain the Hurst exponent as H = slope of the straight line in step 8.

Residuals of Regression Method

1. Divide the sequence into block of size, *p*, that is mutually exclusive
2. Calculate the sum of each block and denote it as, Y_i
3. Fit a least square line of Y_i
4. Evaluate the sample variance of the residuals.
5. Steps 3 and 4, are then repeated for each block.
6. Calculate the average of all the blocks' sample variance
7. Plot a log of Residual Variance vs log of *p* graph
8. Finally, fit the straight line and calculate the Hurst exponent where $h = 0.5(m)$, *m* is the slope

Aggregate Variance Method

1. Divide the price history data into sub-series, *m*.
2. Calculate the mean of each series.
3. Calculate the average of all the series' mean values.
4. Calculate the variance of the series' mean values.
5. Plot a log of Variance vs log of *m* and plot a fitting straight line along with it.
6. Calculate the Hurst exponent where $H = 1 - \frac{m}{2}$

Error Analysis

Finally, after performing the simulations using GBM and GFBM, an error analysis is performed. In this research, the calculation Mean Absolute Percentage Error, MAPE values, are used to evaluate the accuracy of the simulated price. The formula for the calculation is

$$MAPE = \frac{100\%}{n}\sum_{1}^{n}\left|\frac{S_t - SS_t}{S_t}\right|$$

where n is the total number of data, S_t = actual rubber price and SS_t = simulated rubber price

Results and Discussion

This section documents the simulation results for rubber prices using geometric Brownian motion

(GBM) and geometric fractional Brownian motion (GFBM) models, where for the latter, the Hurst exponents are estimated using Hurst estimators, namely, rescaled range analysis method, residuals of regression method, and aggregate variance method.

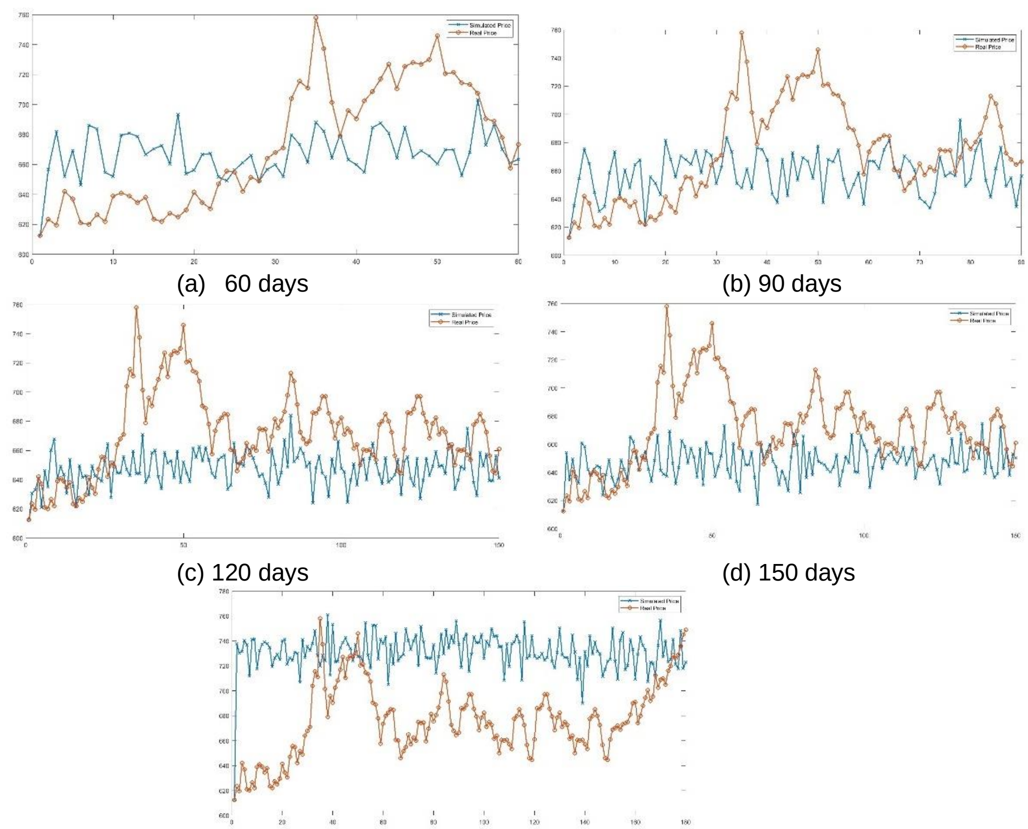
Simulation of Rubber Prices using Geometric Brownian Motion

The simulation of prices using geometric Brownian motion requires the values of mean and standard deviation. Using historical prices, these values are estimated as given in Table 1.

Table 1. The Mean and Standard Deviation values.

Days	Mean	Standard Deviation
60	0.001609136499	0.02036024291
90	0.0009493392944	0.01860613339
120	0.0006403772894	0.01765849183
150	0.0005114422646	0.01717369504
180	0.00112396671	0.01620261934

Figures 2 (a)--(e) plot the actual and simulated prices of the rubber using GBM for 60, 90, 120, 150 and 180 days.



(a) 60 days (b) 90 days

(c) 120 days (d) 150 days

(e) 180 days

Figure 2. Simulation of Rubber Prices using GBM over different time range

Table 2 documents the MAPE values which evidently can be seen that the lowest MAPE value is obtained when we simulated the prices for a period of 120 days, and the value increases as the number of simulations increases to 150 and 180 days.

Table 2. The Error Analysis for the Simulation of Rubber Prices Using Geometric Brownian Motion.

Days	MAPE (%)
60	4.91
90	4.39
120	4.25
150	4.31
180	8.56

Thus, the simulation of the prices using GBM model is more accurate when a smaller size data is used. Other than that, in Figures 2 (b)-(e), we observed disruption of noise, which is defined as information or activity that confuses or misrepresents true underlying patterns that may include tiny price swings and corrections that obscure the broader trend [8].

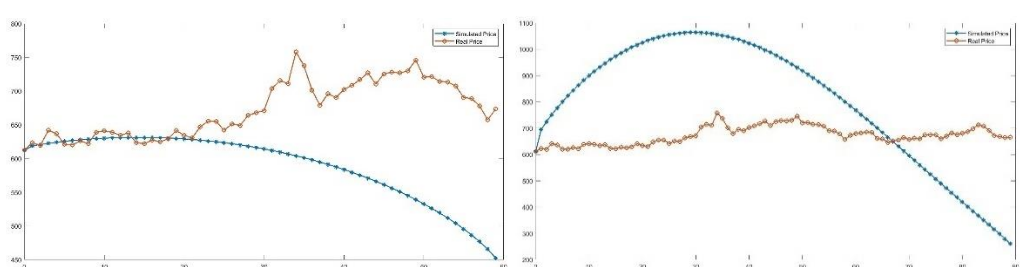
Simulation of Rubber Prices using Geometric Fractional Brownian Motion

To simulate the prices using geometric fractional Brownian motion, we need to estimate the Hurst parameter using the historical prices. We used rescaled range analysis, residuals of regression and aggregate variance methods to estimate the Hurst values over different range of number of days, which are tabulated in Table 3.

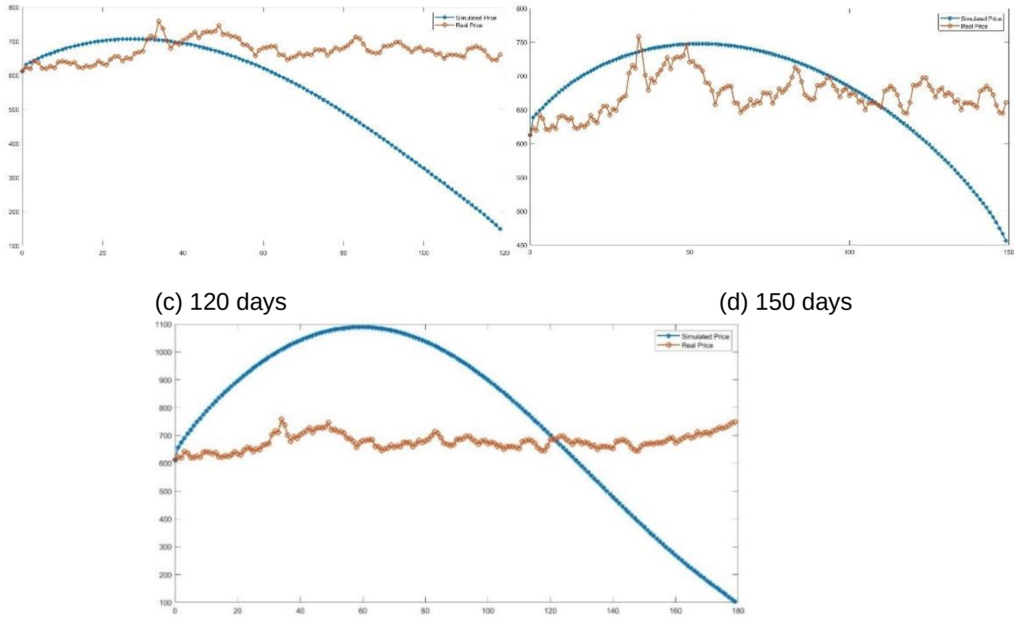
Table 3. Table shows the H, Hurst exponent, obtained for each method and data sequence along with its MAPE Values

Method / Days	60		90		120		150		180	
	H	MAPE (%)	H	MAPE (%)	H	MAPE (%)	H	MAPE (%)	H	MAPE (%)
Rescaled Range Analysis	0.7240	11.91	0.9641	36.10	0.7505	22.80	0.5734	8.60	0.7991	41.46
Residuals of Regression	0.7167	7.65	0.8697	35.53	1.3920	-	1.5389	-	0.8903	52.62
Aggregate Variance	1.0237	-	0.9740	44.60	0.9471	50.21	0.9503	45.30	0.8733	46.92

The Hurst values estimated over a 60-day range using the aggregate variance method, and over a 120-day and 150-day range using the residual of regression is more than 1. According to [12] as cited in [13], this suggests that the data are either non-stationary or the estimation was unsuccessful. Therefore, the MAPE values and the simulation for these were not performed as we chose the Hurst value to lie only in the interval (0,1). We now look at the Hurst values in Table 3, excluding the highlighted values. The rescaled range analysis method estimates the Hurst values for all five ranges, where the lowest MAPE value is 8.60% for a 150-day period with H=0.5734, while the residuals of regression method estimates 3 out of 5 ranges, where the lowest MAPE value is 7.65% for a 60-day period with H=0.7167. The aggregate variance method, on the other hand, estimates 4 out of 5 ranges, where the lowest MAPE value is 44.60% for a 90-day period with H=0.9740.



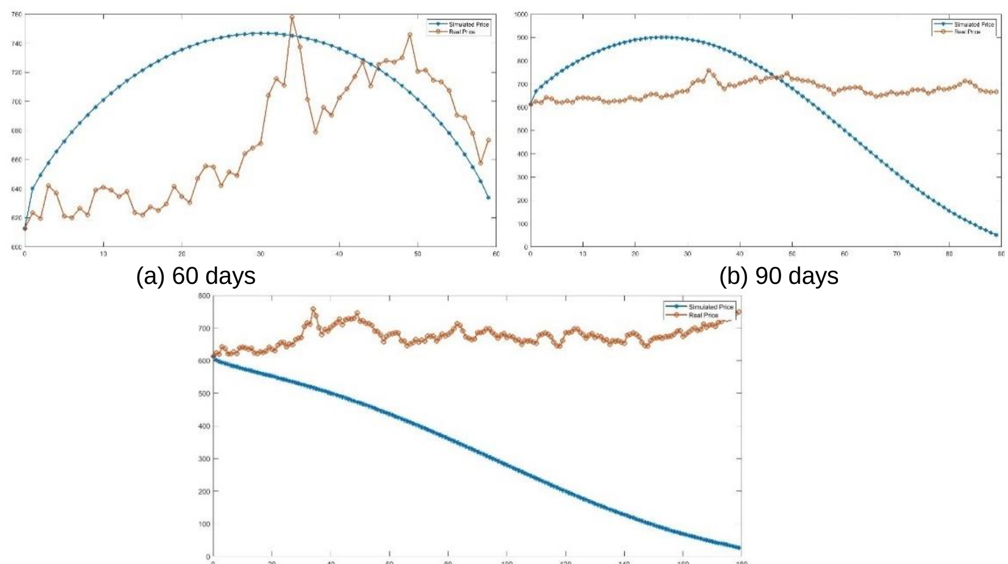
(a) 60 days (b) 90 days



(c) 120 days (d) 150 days

(e) 180 days

Figure 3. Simulation of Rubber Prices with Hurst Values Estimated using Rescaled Range Analysis Method



(a) 60 days (b) 90 days

(c) 180 days

Figure 4. Simulation of Rubber Prices with Hurst Values Estimated using Residuals of Regression Method

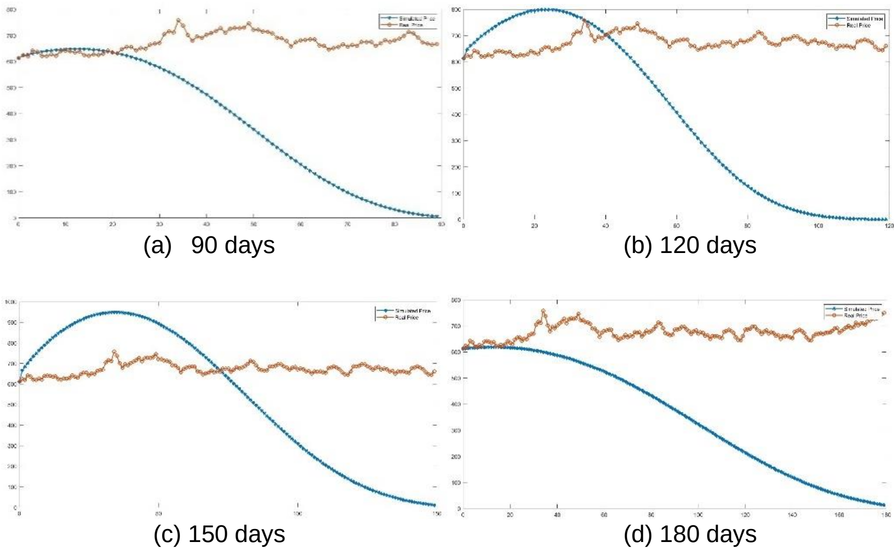


Figure 5. Simulation of Rubber Prices with Hurst Values Estimated using Aggregate Variance Method

Conclusion

In conclusion for the simulation of rubber prices using geometric Brownian motion (GBM), it could clearly be observed that the data sequence of less than 150 has an error less than 5%, whereas, for the data sequence of 180 has a mean absolute percentage error (MAPE) of approximately 9%. Therefore, the smaller data sequence particularly less than 150 provides a more accurate simulation of rubber prices using geometric Brownian motion (GBM). For the simulation of rubber prices using geometric fractional Brownian motion (GFBM) however, without taking account of the different Hurst exponent estimator method, the smaller value of Hurst exponents yields more accurate simulated rubber price.

Future work may include an analysis using bigger size data sequence and analyse more on the long-range dependence as well as analyse on the ways to remove the noise presented in the simulation graph. Moreover, an analysis on how to simulate the commodity prices using GFBM for the non-stationary data may also be done for future purposes as not all the data that is obtained will be stationary.

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References

[1] Malaysia External Trade Statistics Bulletin, June 2021 28 July 2021. [Online]. Available: https://www.dosm.gov.my/v1/index.php. [Accessed 26 December 2022].

[2] E. Valkeila. (2010). On the approximation of geometric fractional brownian motion. *Optimality and Risk - Modern Trends in Mathematical Finance: The Kabanov Festschrift*, Berlin, Heidelberg, Springer Berlin Heidelberg, 251-266.

[3] K. Hamed, A. G. Awadallah, E. Brown, S. Hurst and J. Sutcliffe. (2016). Harold Edwin Hurst: the Nile and Egypt, past and future. *Hydrological Sciences Journal, 61*(9), 1557-1570.

[4] T. Graves, R. Gramacy, N. W. Watkins and C. F. Franzke. (2017). A brief history of long memory: Hurst, Mandelbrot and the road to ARFIMA, 1951-1980, *Entropy, 19*, 437.

[5] S. Aref. (1998). Hurst phenomenon and fractal dimensions in long-term yield data. *Conference on Applied*

Statistics in Agriculture.

[6] B.-R. Cristina-MariaPăcurar. (2020). An analysis of COVID-19 spread based on fractal interpolation and fractal dimension. *Chaos, Solitons & Fractals,139*, 110073.

[7] L. G. Malaysia. Malaysian Rubber Board. [Online]. Available: http://www3.lgm.gov.my/mre/MonthlyPrices.aspx.

[8] C. Chen. (2022). MATLAB Central File Exchange. [Online]. Available: https://www.mathworks.com/matlabcentral/fileexchange/19148-hurst-parameter-estimate. [Accessed 2022].

[9] A. Hibtallah and M. Y. Hmood. (2021). Comparison of Hurst exponent estimation methods. *Journal of Economics and Administrative Sciences, 27*, 167-183.

[10] J. Kuepper. (2022)., Investopedia, July 2022. [Online]. Available: https://www.investopedia.com/articles/trading/06/marketnoise.asp.

[11] R. M. Bryce and K. B. Sprague. (2012). Revisiting detrended fluctuation analysis. *Scientific reports, 2*, 315.

[12] R. F. Ceballos and F. F. Largo. (2017). On the estimation of the hurst exponent using adjusted rescaled range analysis, detrended fluctuation analysis and variance time plot: a case of exponential distribution," *Imperial Journal of Interdisciplinary Research (IJIR), 3*, 424-434.

[13] Z. N. Hamdan, S. N. I. Ibrahim and M. S. Mustafa. (2020). Modelling Malaysian gold prices using geometric brownian motion model. *Advances in Mathematics: Scientific Journal*. *9*, 7463-7469.

[14] S. N. I. Ibrahim, M. Misiran and M. F. Laham. (2020). Geometric fractional Brownian motion model for commodity market simulation. *AEJ - Alexandria Engineering Journal, 60*, 11.

[15] W. F. Agustini, I. R. A. Affianti and E. R. Putri. (2018). Stock price prediction using geometric Brownian motion. *Journal of Physics: Conference Series, 974*, 012047.

[16] M. Y. Hmood and A. Hibtallah. (2021). Comparison of Hurst exponent estimation methods. *Journal of Economics and Administrative Sciences, 27*, 167-183.

[17] D. W. and C. Heyde. (1996). Itô's formula with respect to fractional Brownian motion and its application. *Journal of Applied Mathematics and Stochastic Analysis, 9*, 01.