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RESEARCH ARTICLE

Coefficient Estimate on Second Hankel Determinant of the Logarithmic Coefficients for Close-To-Convex Function Subclass with Respect to the Koebe Function

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Abstract Let S denote the subclass of the analytic function and univalent functions in D, where D is defined as the unit disk and having the Taylor representation form of S. In this paper, we will estimate the second Hankel determinant which the elements are the logarithmic coefficients of the class close-to-convex function with respect to the Koebe function in S.

Keywords: Univalent functions, analytic functions, Hankel determinant, close-to-convex functions.

Introduction

Let S be a subclass of class A, where the class A is analytic and normalized by $f(0)=f'(0)-1=0$ in D. The notation of D is defined as the unit disk, $|z|<1$ such that $z\in C$. If the function $f\in A$, then $f(z)$ has the series form

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n . \tag{1}$$

Starlike functions, convex functions, and close-to-convex functions are the three main subclasses in S. We made S represent the class of univalent functions in A. If the following criteria have been met,

$$\mathrm{Re}\left(\frac{z f'(z)}{f(z)}\right)>0 , \tag{2}$$

then $f\in A$ becomes a starlike function for $z\in D$. The starlike function class is denoted by S^*. It has an important class member, the Koebe function, which can be defined as follows,

$$k(z)=\frac{z}{(1-z)^2} . \tag{3}$$

In the most recent issues for the S^* and S classes, the Koebe function plays a crucial role as an extreme function. When a function $f\in A$ fits the following criteria, it is called a convex function,

$$\mathrm{Re}\left(1+\frac{z f''(z)}{f'(z)}\right)>0 , \tag{4}$$

for $z\in D$. This class is denoted by CV in class S. If there is a real number, α, where $|\alpha|<\pi/2$ and a convex function $g(z)$ that fits the following conditions, a function $f\in A$ is said to be close-to-convex,

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$$\mathrm{Re}\left(e^{i\alpha} \frac{f'(z)}{g'(z)} \right) > 0 , \quad (5)$$

where $z \in D$ [13]. In 1916, Alexander stated that there is a link between a starlike function and a convex function, with the condition that if function $h(z) \in S^*$ then $h(z) = zg'(z)$ and $g(z) \in CV$ respectively [10]. Therefore, the condition (5) can be formed as follows

$$\mathrm{Re}\left(e^{i\alpha} \frac{zf'(z)}{h(z)} \right) > 0 , \quad (6)$$

where $z \in D$. From there, we can deduce that the starlike functions and convex functions are both close-to-convex functions. We may sum it up by $CV \subseteq S^* \subseteq K \subseteq S$ with suitable inclusion. We denote the class of close-to-convex function as K.

The class of S^*, CV and K functions have a representation that uses the Caratheodory class P. The class p is an analytic function P in D by having the following form,

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n , \quad (7)$$

where $z \in D$ and having a positive real part in D. These classes can be expressed by the coefficients of functions in P. The logarithmic coefficients of function f, can be written as,

$$\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n , \quad (8)$$

where $z \in D$. Milin's conjecture relies heavily on the logarithmic coefficients (cited by author in [15]). It can be seen that the class sharp estimates for the single logarithmic coefficient, $|\gamma_1| \leq 1$, $|\gamma_2| \leq 0.635$, and $n = 3,4,\cdots$, are unknown. The study of logarithmic coefficients has recently attracted the public's curiosity. [1], [2], [5], [8], [9], [11], [17], [18], [22] and [23], are just a few examples of logarithmic coefficient estimations that have been established.

Hankel matrices and determinants serve some important roles in mathematics and have a variety of applications. The Hankel determinants for their classes have been identified by many researchers. The authors in [4], [6], [12], [13], [16], [19] and [21] are just a few examples. Recently, the author in [15] issued a sharp finding of the Hankel determinants, whose entries are the logarithmic coefficients of $f \in S$, that is

$$H_{q,n}(F_f/2) = \begin{vmatrix} \gamma_n & \gamma_{n+1} & \cdots & \gamma_{n+q-1} \\ \gamma_{n+1} & \gamma_{n+2} & \cdots & \gamma_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ \gamma_{n+q-1} & \gamma_{n+q} & \cdots & \gamma_{n+2(q-1)} \end{vmatrix},$$

where $q, n \in N$. They are working on determining the Hankel determinants for the starlike and convex function classes in their research. In addition, the author in [3] found a second Hankel determinant with the logarithmic coefficients for the starlike and convex functions classes with excellent results. During that year, the author in [4] obtained an accurate result of the second Hankel determinant of the logarithmic coefficient for some subclasses of the analytic functions.

From there, we were inspired to obtain the Hankel determinant for the subclasses of S, particularly for the class of close-to-convex functions. In the same year, the author in [20] defined the class $K_{\alpha\delta}$ which is a class of close-to-convex function with the satisfaction of the following conditions

$$\mathrm{Re}\left(e^{i\alpha} \frac{z f'(z)}{g(z)} \right) > \delta ,$$

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where $z \in D$, $g(z) \in S^*$, $|\alpha| < \frac{\pi}{2}$, and $\cos(\alpha) > \delta$. From there, we tend to find the Hankel determinant for the class of close-to-convex function K_0 where the function $g(z)$ in the form of Koebe function.

In this paper, we deal with functional $H_{2,1}(F_f/2) = \gamma_1\gamma_3 - \gamma_2^2$ when the function f is close-to-convex univalent function. We know that $H_{2,1}(F_f/2)$ corresponds with the functional $H_{2,1}(f) = a_3 - a_2^2$ over class S or its subclasses. Also, it associated with the coefficient b_1 in the class Σ and area theorem (cited by author in [15]). Besides, this functional for the class S was estimated by Bieberbach in 1916 (cited by author in [15]). We know that the logarithmic coefficients can write as

$$\gamma_1 = \frac{1}{2}a_2, \quad \gamma_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right), \text{ and } \gamma_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right).$$

Therefore,

$$H_{2,1}(F_f/2) = \frac{1}{4}\left(a_2a_4 - a_3^2 + \frac{1}{12}a_2^4\right).$$

Note that, the condition if the function $f \in S$, and $\theta \in R$, then $f_\theta(z) := e^{-i\theta} f\left(e^{-i\theta}z\right)$. Therefore, we have the following inequality,

$$H_{2,1}(F_{f_\theta}/2) = \frac{e^{4i\theta}}{4}\left(a_2a_4 - a_3^2 + \frac{1}{12}a_2^4\right) = e^{4i\theta}H_{2,1}(F_f/2).$$

The objective of this paper is to find the upper bound of $H_{2,1}(F_f/2)$ when the function f is the class of close-to-convex function, K_0. The following Lemmas will be used to get the upper bound of $H_{2,1}(F_f/2)$ for class K_0.

Lemma 1.1. *(Cited by author in [15]) If $p \in \mathrm{P}$ is of the form (7) with $c_1 \geq 0$, then*

$$c_1 = 2\zeta_1, \tag{9}$$

$$c_2 = 2\zeta_1^2 + 2\left(1 - \zeta_1^2\right)\zeta_2, \tag{10}$$

and

$$c_3 = 2\zeta_1^3 + 4\left(1 - \zeta_1^2\right)\zeta_1\zeta_2 - 2\left(1 - \zeta_1^2\right)\zeta_1\zeta_2^2 + 2\left(1 - \zeta_1^2\right)\left(1 - |\zeta_2|^2\right)\zeta_3. \tag{11}$$

for some $\zeta_1 \in [0,1]$ and ζ_2, $\zeta_3 \in \overline{D} := \{z \in C : |z| < 1\}$. For $\zeta_1 \in D$ and $\zeta_2 \in T = \{z \in C : |z| = 1\}$, there is a unique function $p \in \mathrm{P}$ with c_1 and c_2 as in (9)-(10), namely,

$$p(z) = \frac{1 + \left(\bar{\zeta}_1\zeta_2 + \zeta_1\right)z + \zeta_2 z^2}{1 + \left(\bar{\zeta}_1\zeta_2 - \zeta_1\right)z - \zeta_2 z^2}, \quad z \in D.$$

For ζ_1, $\zeta_2 \in D$ and $\zeta_3 \in T$, there is a unique function $p \in \mathrm{P}$ with c_1, c_2 and c_3 as in (9)-(11)

$$p(z) = \frac{1 + \left(\bar{\zeta}_1\zeta_3 + \bar{\zeta}_1\zeta_2 + \zeta_1\right)z + \left(\bar{\zeta}_1\zeta_3 + \zeta_1\bar{\zeta}_2\zeta_3 + \zeta_2\right)z^2 + \zeta_3 z^3}{1 + \left(\bar{\zeta}_1\zeta_3 + \bar{\zeta}_1\zeta_2 - \zeta_1\right)z + \left(\bar{\zeta}_1\zeta_3 - \zeta_1\bar{\zeta}_2\zeta_3 - \zeta_2\right)z^2 - \zeta_3 z^3}, \quad z \in D.$$

Lemma 1.2. *(Cited by author in [3]) Given real numbers A, B, C, let*

$$Y(A,B,C) := \max\left\{\left|A + Bz + Cz^2\right| + 1 - |z|^2 : z \in \overline{D}\right\}$$

I. If $AC \geq 0$, then

$$Y(A,B,C) = \begin{cases} |A| + |B| + |C|, & |B| \geq 2(1 - |C|), \\ 1 + |A| + \dfrac{B^2}{4(1 - |C|)}, & |B| < 2(1 - |C|). \end{cases}$$

II. If $AC < 0$, then

$$Y(A,B,C)=\begin{cases} 1-|A|+\dfrac{B^2}{4(1-|C|)}, & -4AC\left(\dfrac{1}{C^2}-1\right)\le B^2 \wedge |B|<2(1-|C|), \\ 1+|A|+\dfrac{B^2}{4(1-|C|)}, & B^2<\min\left\{4(1+|C|)^2, -4AC\left(\dfrac{1}{C^2}-1\right)\right\} \\ R(A,B,C) & otherwise, \end{cases}$$

where

$$R(A,B,C):=\begin{cases} |A|+|B|-|C|, & |C|(|B|+4|A|)\le |AB|, \\ -|A|+|B|+|C|, & |AB|\le |C|(|B|-4|A|), \\ (|C|+|A|)\sqrt{1-\dfrac{B^2}{4AC}} & otherwise. \end{cases}$$

The following theorem will prove the second Hankel determinant with entries are logarithmic coefficient for the class of close-to-convex function with respect to the Koebe function.

Theorem 1.0. *If the function* $f \in K_0$ *(close-to-convex function), then*

$$\left|\gamma_1\gamma_3-\gamma_2^2\right| \le 0.08513424068$$

The inequality is sharp.

Prove. From the following equation,

$$\frac{z\,f'(z)}{g(z)}=p(z)$$

where $g(z)=K(z)=\dfrac{z}{(1-z)^2}$ is the Koebe function. Then, we have

$$z\,f'(z)=g(z)p(z). \tag{12}$$

By differentiating the equation (12) and computing the coefficients of z^2, z^3, and z^4, we get

$$a_2=1+\frac{1}{2}c_1, \tag{13}$$

$$a_3=\frac{1}{3}c_2+\frac{2}{3}c_1+1, \tag{14}$$

and

$$a_4=\frac{1}{4}c_3+\frac{1}{2}c_2+\frac{3}{4}c_1+1. \tag{15}$$

respectively. Note that, the logarithmic coefficients for γ_1, γ_2, and γ_3 give

$$\gamma_1=\frac{1}{2}a_2, \tag{16}$$

$$\gamma_2=\frac{1}{2}\left(a_3-\frac{1}{2}a_2^2\right), \tag{17}$$

and

$$\gamma_3=\frac{1}{2}\left(a_4-a_2a_3+\frac{1}{3}a_2^3\right). \tag{18}$$

respectively. Then, substitute the equation of (13), (14), and (15) into the logarithmic coefficients, (16) - (18), yields

$$\gamma_1=\frac{1}{2}+\frac{1}{4}c_1, \tag{19}$$

$$\gamma_2=\frac{1}{6}c_2+\frac{1}{12}c_1+\frac{1}{4}-\frac{1}{16}c_1^2, \tag{20}$$

and

$$\gamma_3=\frac{1}{8}c_3+\frac{1}{12}c_2+\frac{1}{24}c_1+\frac{1}{6}-\frac{1}{12}c_1c_2-\frac{1}{24}c_1^2+\frac{1}{48}c_1^3. \tag{21}$$

Then, by applying Lemma 1.1 into the logarithmic coefficient (19) – (20), we have

$$\gamma_1 = \frac{1}{2} + \frac{1}{2}\zeta_1, \tag{22}$$

$$\gamma_2 = \frac{1}{12}\zeta_1^2 + \frac{1}{3}\left(1-\zeta_1^2\right)\zeta_2 + \frac{1}{6}\zeta_1 + \frac{1}{4}, \tag{23}$$

and

$$\gamma_3 = \frac{1}{12}\zeta_1^3 + \frac{1}{6}\left(1-\zeta_1^2\right)\zeta_1\zeta_2 - \frac{1}{4}\left(1-\zeta_1^2\right)\zeta_1\zeta_2^2 + \frac{1}{4}\left(1-\zeta_1^2\right)\left(1-|\zeta_2|^2\right)\zeta_3 + \frac{1}{6}\left(1-\zeta_1^2\right)\zeta_2 + \frac{1}{12}\zeta_1 + \frac{1}{6}. \tag{24}$$

Note that, the second Hankel determinant denotes $H_{2,1}\left(F_f/2\right) = \gamma_1\gamma_3 - \gamma_2^2$. For simplification, we need to let $A_1 = \gamma_1\gamma_3$ and $A_2 = \gamma_2^2$. From there, we get

$$A_1 = \frac{1}{24}\zeta_1^3 + \frac{1}{6}\left(1-\zeta_1^2\right)\zeta_1\zeta_2 - \frac{1}{8}\left(1-\zeta_1^2\right)\zeta_1\zeta_2^2 + \frac{1}{8}(1+\zeta_1)\left(1-\zeta_1^2\right)\left(1-|\zeta_2|^2\right)\zeta_3 + \frac{1}{12}\left(1-\zeta_1^2\right)\zeta_2 + \frac{1}{8}\zeta_1 + \frac{1}{24}\zeta_1^4 + \frac{1}{12}\left(1-\zeta_1^2\right)\zeta_1^2\zeta_2 - \frac{1}{8}\left(1-\zeta_1^2\right)\zeta_1^2\zeta_2^2 + \frac{1}{24}\zeta_1^2$$

and

$$A_2 = \frac{1}{144}\zeta_1^4 + \frac{1}{9}\left(1-\zeta_1^2\right)^2\zeta_2^2 + \frac{5}{72}\zeta_1^2 + \frac{1}{16} + \frac{1}{18}\left(1-\zeta_1^2\right)\zeta_1^2\zeta_2 + \frac{1}{36}\zeta_1^3 + \frac{1}{9}\left(1-\zeta_1^2\right)\zeta_1\zeta_2 + \frac{1}{6}\left(1-\zeta_1^2\right)\zeta_2 + \frac{1}{12}\zeta_1$$

After that, we subtract the equation of A_1 and A_2 ($A_1 - A_2 = \gamma_1\gamma_3 - \gamma_2^2$), and it yields

$$A_1 - A_2 = \frac{1}{72}\zeta_1^3 + \frac{1}{24}\zeta_1 + \frac{5}{144}\zeta_1^4 - \frac{1}{36}\zeta_1^2 + \frac{1}{48} + \zeta_2\left(1-\zeta_1^2\right)\left(\frac{1}{18}\zeta_1 - \frac{1}{12} + \frac{1}{36}\zeta_1^2\right) + \zeta_2^2\left(1-\zeta_1^2\right)\left(-\frac{1}{8}\zeta_1 - \frac{1}{8}\zeta_1^2 - \frac{1}{9}\left(1-\zeta_1^2\right)\right) + \frac{1}{8}(1+\zeta_1)\left(1-\zeta_1^2\right)\left(1-|\zeta_2|^2\right)\zeta_3$$

and gives

$$A_1 - A_2 = \frac{1}{72}\zeta_1^3 + \frac{1}{24}\zeta_1 + \frac{5}{144}\zeta_1^4 - \frac{1}{36}\zeta_1^2 + \frac{1}{48} + \frac{1}{36}\left(1-\zeta_1^2\right)(\zeta_1+3)(\zeta_1-1)\zeta_2 - \frac{1}{72}\left(1-\zeta_1^2\right)(\zeta_1+8)(\zeta_1+1)\zeta_2^2 + \frac{1}{8}(1+\zeta_1)\left(1-\zeta_1^2\right)\left(1-|\zeta_2|^2\right)\zeta_3$$

A). Suppose that $\zeta_1 = 1$, then

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| \le \left|\frac{1}{72}(1)^3 + \frac{1}{24}(1) + \frac{5}{144}(1)^4 - \frac{1}{36}(1)^2 + \frac{1}{48} + \frac{1}{36}\left(1-(1)^2\right)((1)+3)((1)-1)\zeta_2 - \frac{1}{72}\left(1-(1)^2\right)((1)+8)((1)+1)\zeta_2^2 + \frac{1}{8}(1+(1))\left(1-(1)^2\right)\left(1-|\zeta_2|^2\right)\zeta_3\right|$$

$$= \left|\frac{1}{72} + \frac{1}{24} + \frac{5}{144} - \frac{1}{36} + \frac{1}{48}\right|$$

$$= \frac{1}{12} \approx 0.083333$$

B). Suppose that $\zeta_1 = 0$, and $|\zeta_2| \le 1$ then

$$\left|\gamma_1\gamma_3 - \gamma_2^2\right| \le \left|\frac{1}{72}(0)^3 + \frac{1}{24}(0) + \frac{5}{144}(0)^4 - \frac{1}{36}(0)^2 + \frac{1}{48} + \frac{1}{36}\left(1-(0)^2\right)((0)+3)((0)-1)\zeta_2 - \frac{1}{72}\left(1-(0)^2\right)((0)+8)((0)+1)\zeta_2^2 + \frac{1}{8}(1+(0))\left(1-(0)^2\right)\left(1-|\zeta_2|^2\right)\zeta_3\right|$$

$$= \left|\frac{1}{48} + \frac{1}{36}(3)(-1)\zeta_2 - \frac{1}{72}(8)\zeta_2^2 + \frac{1}{8}\left(1-|\zeta_2|^2\right)\zeta_3\right|$$

$$= \frac{31}{144} \approx 0.21527$$

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C). Suppose that $\zeta_1 \in (0,1)$, Since $|\zeta_3| \le 1$, we have

$$\gamma_1\gamma_3 - \gamma_2^2 = \frac{1}{72}\zeta_1^3 + \frac{1}{24}\zeta_1 + \frac{5}{144}\zeta_1^4 - \frac{1}{36}\zeta_1^2 + \frac{1}{48} + \frac{1}{36}(1-\zeta_1^2)(\zeta_1+3)(\zeta_1-1)\zeta_2 - \frac{1}{72}(1-\zeta_1^2)(\zeta_1+8)(\zeta_1+1)\zeta_2^2 + \frac{1}{8}(1+\zeta_1)(1-\zeta_1^2)(1-|\zeta_2|^2)$$

and gives

$$\gamma_1\gamma_3 - \gamma_2^2 = \frac{1}{8}(1+\zeta_1)(1-\zeta_1^2)\left[\frac{2}{9(1+\zeta_1)} + \frac{1}{(1+\zeta_1)}\left(\frac{2\zeta_1^2}{9} + \frac{4\zeta_1}{9} - \frac{2}{3}\right)\zeta_2 + \frac{1}{(1+\zeta_1)}\left(-\frac{\zeta_1^2}{9} - \zeta_1 - \frac{8}{9}\right)\zeta_2^2 + 1 - |\zeta_2|^2\right] + \frac{5\zeta_1^4}{144} + \frac{\zeta_1^3}{72} + \frac{\zeta_1}{24} - \frac{1}{144}.$$

Then, we have

$$|\gamma_1\gamma_3 - \gamma_2^2| = \frac{1}{8}(1+\zeta_1)(1-\zeta_1^2)\left[\left|A + B\zeta_2 + C\zeta_2^2\right| + 1 - |\zeta_2|^2\right] + \left|\frac{5\zeta_1^4}{144} + \frac{\zeta_1^3}{72} + \frac{\zeta_1}{24} - \frac{1}{144}\right|, \tag{25}$$

where,

$$A = \frac{2}{9(1+\zeta_1)}, \quad B = \frac{1}{(1+\zeta_1)}\left(\frac{2\zeta_1^2}{9} + \frac{4\zeta_1}{9} - \frac{2}{3}\right) \text{ and } C = \frac{1}{(1+\zeta_1)}\left(-\frac{\zeta_1^2}{9} - \zeta_1 - \frac{8}{9}\right)$$

We tend to solve the equation $(1/8)(1+\zeta_1)(1-\zeta_1^2)\left[\left|A + B\zeta_2 + C\zeta_2^2\right| + 1 - |\zeta_2|^2\right]$ by using the lemma 1.2, then we solve it together with the rest of the equation. The first step is showing the inequality of $AC < 0$, for $\zeta_1 \in (0,1)$, that gives

$$\frac{2}{9(1+\zeta_1)^2}\left(-\frac{\zeta_1^2}{9} - \zeta_1 - \frac{8}{9}\right) < 0,$$

and further simplification, we have the following inequality,

$$-\frac{(2\zeta_1 + 16)}{81(1+\zeta_1)} < 0,$$

which is showed decreasing for $\zeta_1 \in (0,1)$. Since $AC < 0$, we apply Lemma 1.2 only for case II. Before we venture into more details, we know that the absolutes of A, B and C give

$$|A| = 2\left[\frac{1}{(9+9\zeta_1)^2}\right]^{\frac{1}{2}}, \quad |B| = \left[\frac{(2\zeta_1^2 + 4\zeta_1 - 6)^2}{(9+9\zeta_1)^2}\right]^{\frac{1}{2}}, \text{ and } |C| = \left(\frac{(\zeta_1+8)^2}{9}\right)^{\frac{1}{2}}.$$

C1). Note that, the inequality of $-4AC\left(\frac{1}{C^2} - 1\right) - B^2 \le 0$, gives

$$-\frac{8\left(-\frac{\zeta_1^2}{9} - \zeta_1 - \frac{8}{9}\right)\left(\frac{(1+\zeta_1)^2}{\left(-\frac{\zeta_1^2}{9} - \zeta_1 - \frac{8}{9}\right)^2} - 1\right)}{(9+9\zeta_1)(1+\zeta_1)} - \frac{\left(\frac{2\zeta_1^2}{9} - \frac{4\zeta_1}{9} - \frac{2}{3}\right)^2}{(1+\zeta_1)^2} \le 0$$

Further simplification of the above equation, we have

$$\frac{-4\zeta_1^5 - 48\zeta_1^4 - 128\zeta_1^3 - 24\zeta_1^2 + 356\zeta_1 - 152}{81(1+\zeta_1)^2(\zeta_1+8)} \le 0$$

yields

$$-4\zeta_1^5 - 48\zeta_1^4 - 128\zeta_1^3 - 24\zeta_1^2 + 356\zeta_1 - 152 \le 0$$

By solving it computationally using the Maple software of the 2019 version for the equation, we have the following roots $\zeta_1 \approx -8.453204077$, $\zeta_1 \approx 0.4958717766$ and $\zeta_1 \approx 1$. The root for $\zeta_1 \approx 0.4958717766$ and $\zeta_1 \approx 1$ are hold for $\zeta_1 \in (0,1)$ of the inequality $-4\zeta_1^5 - 48\zeta_1^4 - 128\zeta_1^3 - 24\zeta_1^2 + 356\zeta_1 - 152 \leq 0$. Now we look into another inequality for $|B| - 2(1 - |C|) < 0$, which gives

$$2\left[\frac{\zeta_1^4}{(9+9\zeta_1)^2} + \frac{4\zeta_1^3}{(9+9\zeta_1)^2} - \frac{2\zeta_1^2}{(9+\zeta_1)^2} - \frac{12\zeta_1}{(9+9\zeta_1)^2} + \frac{9}{(9+9\zeta_1)^2}\right]^{\frac{1}{2}} - 2 + \frac{2}{9}\left[(\zeta_1+8)^2\right]^{\frac{1}{2}} < 0,$$

and

$$\frac{2}{(9+9\zeta_1)}\left(\zeta_1^2 + 2\zeta_1 - 3\right) - 2 + \frac{2}{9}(\zeta_1+8) < 0.$$

By simplifying the equation, we get

$$\frac{4\zeta_1^2 + 4\zeta_1 - 8}{(9+9\zeta_1)} < 0,$$

and

$$4\zeta_1^2 + 4\zeta_1 - 8 < 0,$$

which is false for $\zeta_1 \in (0,1)$. Therefore, for the case C1 is not fulfilled

C2). Next, we move on to the inequality of $B^2 < \min\left\{4(1+|C|)^2, -4AC\left(\frac{1}{C^2}-1\right)\right\}$. From there, we have

$$4(1+|C|)^2 = 4\left(1 + \frac{\sqrt{(\zeta_1+8)^2}}{9}\right)^2,$$

which gives

$$4(1+|C|)^2 = \frac{580}{81} - \frac{(8)\sqrt{\zeta_1^2 + 16\zeta_1 + 64}}{9} + \frac{4\zeta_1^2}{81} + \frac{64\zeta_1}{81}.$$

Next, the following equation can be expressed as

$$-4AC\left(\frac{1}{C^2}-1\right) = \frac{8\left(-\frac{\zeta_1^2}{9} - \zeta_1 - \frac{8}{9}\right)\left(\frac{(1+\zeta_1)^2}{\left(-\frac{\zeta_1^2}{9} - \zeta_1 - \frac{8}{9}\right)^2} - 1\right)}{(9+9\zeta_1)(1+\zeta_1)},$$

and by further simplify it, we get

$$-4AC\left(\frac{1}{C^2}-1\right) = \frac{-8\zeta_1^2 - 128\zeta_1 + 136}{81\zeta_1^2 + 729\zeta_1 + 648}.$$

By substituting any values for $\zeta_1 \in (0,1)$, and comparing the solution for the following equation,

$$\frac{580}{81} - \frac{(8)\sqrt{\zeta_1^2 + 16\zeta_1 + 64}}{9} + \frac{4\zeta_1^2}{81} + \frac{64\zeta_1}{81}$$

and

$$\frac{-8\zeta_1^2 - 128\zeta_1 + 136}{81\zeta_1^2 + 729\zeta_1 + 648}.$$

From there, we know that,

$$4(1+|C|)^2 < -4AC\left(\frac{1}{C^2}-1\right),$$

Then, we can say that $B^2 < 4(1+|C|)^2$. From that inequality, we know that $B^2 - 4(1+|C|)^2 < 0$, and gives

$$\frac{\left(\frac{2\zeta_1^2}{9}+\frac{4\zeta_1}{9}-\frac{2}{3}\right)^2}{(1+\zeta_1)^2}-4\left(1-\frac{\sqrt{(\zeta_1+8)^2}}{9}\right)^2<0,$$

and further simplification, we get

$$\frac{16\left(\zeta_1^3-3\zeta_1+2\right)}{81(1+\zeta_1)^2}<0$$

By solving computationally of the inequality above, we noted that the roots obtained are $\zeta_1\approx 1$ and $\zeta_1\approx -2$. These do not lie on the interval $(0,1)$ for ζ_1. Therefore, the case C2 is unfulfilled.

C3). Now, we look on the inequality of $|C|\left(|B|+4|A|\right)\leq |AB|$. Before that, we need to calculate them separately for a better simplification. We know that

$$AB=\frac{4\left(\zeta_1^2+2\zeta_1-3\right)}{81\,(1+\zeta_1)^2} \text{ and } |AB|=\frac{4\left(\frac{(\zeta_1^2+2\zeta_1-3)^2}{(1+\zeta_1)^4}\right)^{\frac{1}{2}}}{81}.$$

Then, the inequality of $|C|\left(|B|+4|A|\right)-|AB|\leq 0$, can be written as

$$\frac{\left(\sqrt{(\zeta_1+8)^2}\right)\left(\sqrt{\frac{(2\zeta_1^2+4\zeta_1-6)^2}{(9+9\zeta_1)^2}}+(8)\left(\sqrt{\frac{1}{(9+9\zeta_1)^2}}\right)\right)}{9}+\frac{(4)\left(\sqrt{\frac{(\zeta_1^2+2\zeta_1-3)^2}{(1+\zeta_1)^4}}\right)}{81}\leq 0,$$

and

$$\frac{(\zeta_1+8)\left(\frac{2\zeta_1^2+4\zeta_1-6}{9+9\zeta_1}+\frac{8}{9+9\zeta_1}\right)}{9}+\frac{(4)\left(\frac{\zeta_1^2+2\zeta_1-3}{(1+\zeta_1)^2}\right)}{81}\leq 0.$$

We can further simplify the inequality as follows,

$$\frac{2\zeta_1^4++22\zeta_1^3+50\zeta_1^2+42\zeta_1+28}{81(1+\zeta_1)^2}\leq 0$$

From there, we can see that the obtained roots for the above inequality, yielding

$$\zeta_1=-2 \text{ and } \zeta_1=-\frac{\left(540+60\sqrt{21}\right)^{\frac{1}{3}}}{3}-\frac{20}{\left(540+60\sqrt{21}\right)^{\frac{1}{3}}}-3\approx -8.254730169,$$

which the roots are not satisfied for $\zeta_1\in(0,1)$. Therefore, the case C3 is failed.

C4). Next, we look into the inequality of $|AB|\leq |C|\left(|B|-4|A|\right)$, that gives

$$\frac{(4)\left(\sqrt{\frac{(\zeta_1^2+2\zeta_1-3)^2}{(1+\zeta_1)^4}}\right)}{81}-\frac{\left(\sqrt{(\zeta_1+8)^2}\right)\left(\sqrt{\frac{(2\zeta_1^2+4\zeta_1-6)^2}{(9+9\zeta_1)^2}}-(8)\left(\sqrt{\frac{1}{(9+9\zeta_1)^2}}\right)\right)}{9}\leq 0,$$

and.

$$\frac{(4)\left(\frac{\zeta_1^2+2\zeta_1-3}{(1+\zeta_1)^2}\right)}{81}-\frac{(\zeta_1+8)\left(\frac{2\zeta_1^2+4\zeta_1-6}{9+9\zeta_1}-\frac{8}{9+9\zeta_1}\right)}{9}\leq 0.$$

We can simplify the above inequality becomes,

$$-\frac{2\left(\zeta_1^4-11\zeta_1^3+17\zeta_1^2-51\zeta_1-50\right)}{81(1+\zeta_1)^2}\leq 0.$$

10.11113/mjfas.v19n2.2675

Then, we obtained the root for the inequality as follows,

$$\zeta_1 \approx -0.8602810742,$$

which the root obtained do not lie on the interval $(0,1)$ for ζ_1. Therefore, the case C4 is not satisfied.

C5). Lastly, we choose the inequality of $(|C|+|A|)\sqrt{1-\frac{B^2}{4AC}}$. Then

$$|\gamma_1\gamma_3-\gamma_2^2| \le \frac{1}{8}(1+\zeta_1)(1-\zeta_1^2)(|C|+|A|)\sqrt{1-\frac{B^2}{4AC}}+\left|\frac{5\zeta_1^4}{144}+\frac{\zeta_1^3}{72}+\frac{\zeta_1}{24}-\frac{1}{144}\right|$$

which gives,

$$|\gamma_1\gamma_3-\gamma_2^2| \le \frac{1}{32}(1+\zeta_1)(1-\zeta_1^2)\left(\frac{\sqrt{(\zeta_1+8)^2}}{9}+2\left(\sqrt{\frac{1}{(9+9\zeta_1)^2}}\right)\right)\left(\sqrt{16-\frac{2\left(\frac{2\zeta_1^2}{9}+\frac{4\zeta_1}{9}-\frac{2}{3}\right)^2(9+9\zeta_1)}{(1+\zeta_1)\left(-\frac{\zeta_1^2}{9}-\zeta_1-\frac{8}{9}\right)}}\right)$$

$$+\frac{5\zeta_1^4}{144}+\frac{\zeta_1^3}{72}+\frac{\zeta_1}{24}+\frac{1}{144}:=\psi(\zeta_1)$$

Now, we determine in which of the half-plane (left or right) that provide the root on the interval $(0,1)$ for the function $\psi(\zeta_1)$. First, we observe the function $\psi(\zeta_1)$ for the right half-plane, which gives

$$\psi(\zeta_1)=\frac{1}{32}(1+\zeta_1)(1-\zeta_1^2)\left(\frac{(\zeta_1+8)}{9}+\frac{2}{(9+9\zeta_1)}\right)\left(\sqrt{16-\frac{2\left(\frac{2\zeta_1^2}{9}+\frac{4\zeta_1}{9}-\frac{2}{3}\right)^2(9+9\zeta_1)}{(1+\zeta_1)\left(-\frac{\zeta_1^2}{9}-\zeta_1-\frac{8}{9}\right)}}\right)$$

$$+\frac{5\zeta_1^4}{144}+\frac{\zeta_1^3}{72}+\frac{\zeta_1}{24}+\frac{1}{144}.$$

By solving computationally for the numerical root finding, we obtained the roots for $\zeta_1 \approx 1.005506079$ which is false for $\zeta_1 \in (0,1)$. Again, we used the same approach to obtain the root for the left half plane for the function $\psi(\zeta_1)$. We have

$$\psi(\zeta_1)=\frac{1}{32}(1+\zeta_1)(1-\zeta_1^2)\left(\frac{(-1)(\zeta_1+8)}{9}+\frac{(-2)}{(9+9\zeta_1)}\right)\left(\sqrt{16-\frac{2\left(\frac{2\zeta_1^2}{9}+\frac{4\zeta_1}{9}-\frac{2}{3}\right)^2(9+9\zeta_1)}{(1+\zeta_1)\left(-\frac{\zeta_1^2}{9}-\zeta_1-\frac{8}{9}\right)}}\right)$$

$$+\frac{5\zeta_1^4}{144}+\frac{\zeta_1^3}{72}+\frac{\zeta_1}{24}+\frac{1}{144},$$

which gives the root $\zeta_1 \approx 0.9945678421$, and this root holds for $\zeta_1 \in (0,1)$. Therefore, we have,

$$|\gamma_1\gamma_3-\gamma_2^2| \le \psi(0.9945678421) \approx 0.08513424$$

This concludes the proof.

Conclusions

We summarised the inequality in (25) that followed from the section A until C, the equality for the function $f \in A$ provided by equation (12), where $p \in P$. It has the form of (7) with $\zeta_1 \approx 0.9945678421$, $\zeta_2 \approx 1$, and $\zeta_3 \approx 1$ that give,

$$\frac{z\,f'(z)}{g(z)}=\frac{1+(1.989135684)z+z^2}{1-z^2},\quad z \in D$$

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Conflicts of Interest

The author(s) declare(s) that there is no conflict of interest regarding the publication of this paper.

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