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Solutions to the Diophantine Equation $x^2 + 16 \cdot 7^b = y^{2r}$

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Abstract We present a method of determining integral solutions to the equation $x^2 + 16 \cdot 7^b = y^{2r}$, where $x, y, b, r \in \mathbb{Z}^+$. We observe that the results can be classified into several categories. Under each category, a general formula is obtained using the geometric progression method. We then provide the bound for the number of non-negative integral solutions associated with each b. Lastly, the general formula for each of the categories is obtained and presented to determine the respective values of x and y^r. We also highlight two special cases where different formulae are needed to represent their integral solutions.

Keywords: Diophantine equation, integral solution, geometric progression, polynomial.

Introduction

A Diophantine equation is an equation in which only integer solutions are allowed. The main objective of studying Diophantine equations is to determine whether a solution exists. If it does exist, how many solutions are there and can one provide a general form to represent the solutions. Note that Fermat's Last Theorem has been one of the most popular problems since 1637, and the proof was published successfully by Wiles (1995) in the 20th century.

A considerable amount of literature related to Diophantine equations has been published over the years. The Diophantine equation

$$x^2 + C = y^n, \qquad (1)$$

where $x, y \geq 1$ and $n \geq 3$ was investigated since 1850. A survey of Eq. (1) can be found in Abu Muriefah & Bugeaud (2006). Lebesgue (1850) first proved that there exist no non-trivial solutions for $C = 1$. The problem was then extended by replacing the value of C using many other values (Ko, 1965; Cohn, 1993; Mignotte & Weger, 1996). In recent years, Eq. (1) has been studied in different forms by replacing the constant C with a power of a fixed prime p^k. For $p = 2$, Cohn (1992) showed that if k is odd and $n = 3$, there are exactly three families of solutions. However, the situation becomes more complex when k is even (Cohn, 1999). Arif and Abu Muriefah (1998) then investigated the case when $p = 3$ and k is odd, and showed that there is exactly one family of solutions. The case when $p = 3$ and k is even was solved by Luca (2000) under the assumption that $\gcd(x, y) = 1$.

Later, Eq. (1) was also extended by replacing C with two fixed primes that have arbitrary non-negative exponents. By assuming that x and y are coprime, Luca (2002) solved the case for $C = 2^a 3^b$, where a and b are both non-negative integers. The case when $C = 2^a 5^b$ was solved by Luca and Togbe (2008) six years later. Equation (1) was then investigated by replacing $C = 2^a 7^b$. When n is even, Yow (2011) found that there exist infinitely many solutions to the equation. The generalisations for the cases when $a = 2$ and $a = 3$ can be found in Yow et al. (2013) and Sapar et al. (2021), respectively.

In this article, we aim to determine general formulae that give integral solutions to the equation $x^2 + 16 \cdot 7^b = y^{2r}$. The main result is as follows:

Theorem 1. *Let $b, r, t \in \mathbb{Z}^+$. The generators $\left(x_{b,i}, y_{b,i}^r\right)$ of solutions to the equation $x^2 + 16 \cdot 7^b = y^{2r}$ have the following forms:*
when $i = 3t - 2$,

$$x_{b,i} = 7^{\frac{1}{3}i - \frac{1}{3}}\left(2^2 \cdot 7^{b - \frac{2}{3}i + \frac{2}{3}} - 1\right),$$
$$y_{b,i}^r = 7^{\frac{1}{3}i - \frac{1}{3}}\left(2^2 \cdot 7^{b - \frac{2}{3}i + \frac{2}{3}} + 1\right),$$

when $i = 3t - 1$,

$$x_{b,i} = 7^{\frac{1}{3}i - \frac{2}{3}}\left(2 \cdot 7^{b - \frac{2}{3}i + \frac{4}{3}} - 2\right),$$
$$y_{b,i}^r = 7^{\frac{1}{3}i - \frac{2}{3}}\left(2 \cdot 7^{b - \frac{2}{3}i + \frac{4}{3}} + 2\right),$$

when $i = 3t$,

$$x_{b,i} = 7^{\frac{1}{3}i - 1}\left(7^{b - \frac{2}{3}i + 2} - 2^2\right),$$
$$y_{b,i}^r = 7^{\frac{1}{3}i - 1}\left(7^{b - \frac{2}{3}i + 2} + 2^2\right),$$

where i is the ith set of non-negative integral solutions associated with each b.

The details of the proof are given in the following section. We also determine the bound for the number of non-negative integral solutions associated with each b, discussing two special cases of the equation.

Proof of the Main Theorem

In this section, we give the details of the proof of Theorem 1. We first have the following definition and lemma.

Definition 1. *Let $b, r \in \mathbb{Z}^+$. The pair of integers (x, y^r) is a* generator *of solutions to the equation $x^2 + 16 \cdot 7^b = y^{2r}$.*

Lemma 2 (Yow et al., 2013). *Let $a, b, r \in \mathbb{Z}^+$ and $r > 1$. The generators of solutions to the equation $x^2 + 2^a \cdot 7^b = y^{2r}$ are given by*

$$x = 2^{a-p-1} \cdot 7^{b-q} - 2^{p-1} \cdot 7^q,$$
$$y^r = 2^{a-p-1} \cdot 7^{b-q} + 2^{p-1} \cdot 7^q,$$

or

$$x = 2^{a-p-1} \cdot 7^q - 2^{p-1} \cdot 7^{b-q},$$
$$y^r = 2^{a-p-1} \cdot 7^q + 2^{p-1} \cdot 7^{b-q},$$

where $0 < p < a$ and $0 \le q \le b$. □

We now prove Theorem 1 by determining the generators of solutions to the equation $x^2 + 16 \cdot 7^b = y^{2r}$ under three scenarios.

Proof of Theorem 1. Let $b, r \in \mathbb{Z}^+$ and i be the ith set of non-negative integral solutions associated with each b.

By substituting $a = 4$ into the generators in Lemma 2, we obtain

$$x = 2^{3-p} \cdot 7^{b-q} - 2^{p-1} \cdot 7^q, \tag{2}$$
$$y^r = 2^{3-p} \cdot 7^{b-q} + 2^{p-1} \cdot 7^q, \tag{3}$$

where $0 < p < 4$ and $0 \le q \le b$. Note that if we fix the value of b and substitute all possible values of p and q into Eq. (2), the value of x could either be positive or negative. Since we are only interested in non-negative integral solutions, we omit all the generators that contain a negative x value from now onwards.

Recall that the value of i represents the ith set of solutions. By substituting the relevant values for each variable into Eqs. (2) and (3), and listing down all the non-negative integral solutions in descending order, we obtain the following sets of solutions.

When $b = 1$,

i	$x_{b,i}$	$y_{b,i}^r$
1	27	29
2	12	16
3	3	11

When $b = 2$,

i	$x_{b,i}$	$y_{b,i}^r$
1	$195 = 27(7) + 6$	$197 = 29(7) - 6$
2	$96 = 12(7) + 12$	$100 = 16(7) - 12$
3	$45 = 3(7) + 24$	$53 = 11(7) - 24$
4	$21 = 3(7)$	$35 = 5(7)$
5	0	$28 = 4(7)$

When $b = 3$,

i	$x_{b,i}$	$y_{b,i}^r$
1	$1371 = 27(7^2) + 6(7) + 6$	$1373 = 29(7^2) - 6(7) - 6$
2	$684 = 12(7^2) + 12(7) + 12$	$688 = 16(7^2) - 12(7) - 12$
3	$339 = 3(7^2) + 24(7) + 24$	$347 = 11(7^2) - 24(7) - 24$
4	$189 = 27(7)$	$203 = 29(7)$
5	$84 = 12(7)$	$112 = 16(7)$
6	$21 = 3(7)$	$77 = 11(7)$

By using the same approach, other values of $x_{b,i}$ and $y_{b,i}^r$ are obtained as follows.

When $b = 4$,

$x_{4,1} = 9603 = 27(7^3) + 6(7^2) + 6(7) + 6$
$y_{4,1}^r = 9605 = 29(7^3) - 6(7^2) - 6(7) - 6$
$x_{4,2} = 4800 = 12(7^3) + 12(7^2) + 12(7) + 12$
$y_{4,2}^r = 4804 = 16(7^3) - 12(7^2) - 12(7) - 12$
$x_{4,3} = 2397 = 3(7^3) + 24(7^2) + 24(7) + 24$
$y_{4,3}^r = 2405 = 11(7^3) - 24(7^2) - 24(7) - 24$
$x_{4,4} = 1365 = 27(7^2) + 6(7)$
$y_{4,4}^r = 1379 = 29(7^2) - 6(7)$
$x_{4,5} = 672 = 12(7^2) + 12(7)$
$y_{4,5}^r = 700 = 16(7^2) - 12(7)$
$x_{4,6} = 315 = 3(7^2) + 24(7)$
$y_{4,6}^r = 371 = 11(7^2) - 24(7)$
$x_{4,7} = 147 = 3(7^2)$
$y_{4,7}^r = 245 = 5(7^2)$
$x_{4,8} = 0$
$y_{4,8}^r = 196 = 4(7^2)$

When $b = 5$,

$x_{5,1} = 67227 = 27(7^4) + 6(7^3) + 6(7^2) + 6(7) + 6$
$y_{5,1}^r = 67229 = 29(7^4) - 6(7^3) - 6(7^2) - 6(7) - 6$
$x_{5,2} = 33612 = 12(7^4) + 12(7^3) + 12(7^2) + 12(7) + 12$
$y_{5,2}^r = 33616 = 16(7^4) - 12(7^3) - 12(7^2) - 12(7) - 12$
$x_{5,3} = 16803 = 3(7^4) + 24(7^3) + 24(7^2) + 24(7) + 24$
$y_{5,3}^r = 16811 = 11(7^4) - 24(7^3) - 24(7^2) - 24(7) - 24$
$x_{5,4} = 9597 = 27(7^3) + 6(7^2) + 6(7)$
$y_{5,4}^r = 9611 = 29(7^3) - 6(7^2) - 6(7)$
$x_{5,5} = 4788 = 12(7^3) + 12(7^2) + 12(7)$
$y_{5,5}^r = 4816 = 16(7^3) - 12(7^2) - 12(7)$
$x_{5,6} = 2373 = 3(7^3) + 24(7^2) + 24(7)$
$y_{5,6}^r = 2429 = 11(7^3) - 24(7^2) - 24(7)$
$x_{5,7} = 1323 = 27(7^2)$
$y_{5,7}^r = 1421 = 29(7^2)$
$x_{5,8} = 588 = 12(7^2)$

$$y^r_{5,8} = 784 = 16(7^2)$$
$$x_{5,9} = 147 = 3(7^2)$$
$$y^r_{5,9} = 539 = 11(7^2)$$

When $b = 6$,

$$x_{6,1} = 470595 = 27(7^5) + 6(7^4) + 6(7^3) + 6(7^2) + 6(7) + 6$$
$$y^r_{6,1} = 470597 = 29(7^5) - 6(7^4) - 6(7^3) - 6(7^2) - 6(7) - 6$$
$$x_{6,2} = 235296 = 12(7^5) + 12(7^4) + 12(7^3) + 12(7^2) + 12(7) + 12$$
$$y^r_{6,2} = 235300 = 16(7^5) - 12(7^4) - 12(7^3) - 12(7^2) - 12(7) - 12$$
$$x_{6,3} = 117645 = 3(7^5) + 24(7^4) + 24(7^3) + 24(7^2) + 24(7) + 24$$
$$y^r_{6,3} = 117653 = 11(7^5) - 24(7^4) - 24(7^3) - 24(7^2) - 24(7) - 24$$
$$x_{6,4} = 67221 = 27(7^4) + 6(7^3) + 6(7^2) + 6(7)$$
$$y^r_{6,4} = 67235 = 29(7^4) - 6(7^3) - 6(7^2) - 6(7)$$
$$x_{6,5} = 33600 = 12(7^4) + 12(7^3) + 12(7^2) + 12(7)$$
$$y^r_{6,5} = 33628 = 16(7^4) - 12(7^3) - 12(7^2) - 12(7)$$
$$x_{6,6} = 16779 = 3(7^4) + 24(7^3) + 24(7^2) + 24(7)$$
$$y^r_{6,6} = 16835 = 11(7^4) - 24(7^3) - 24(7^2) - 24(7)$$
$$x_{6,7} = 9555 = 27(7^3) + 6(7^2)$$
$$y^r_{6,7} = 9653 = 29(7^3) - 6(7^2)$$
$$x_{6,8} = 4704 = 12(7^3) + 12(7^2)$$
$$y^r_{6,8} = 4900 = 16(7^3) - 12(7^2)$$
$$x_{6,9} = 2205 = 3(7^3) + 24(7^2)$$
$$y^r_{6,9} = 2597 = 11(7^3) - 24(7^2)$$
$$x_{6,10} = 1029 = 3(7^3)$$
$$y^r_{6,10} = 1715 = 5(7^3)$$
$$x_{6,11} = 0$$
$$y^r_{6,11} = 1372 = 4(7^3)$$

For simplicity, we omit the sets of solutions when $b > 6$. We now provide the general forms of generators $x_{b,i}$ and $y^r_{b,i}$ to the equation $x^2 + 16 \cdot 7^b = y^{2r}$. Using the above empirical results, we group the generators based on the value of i. Hence, we have

$$x_{1,1} = 27$$
$$x_{2,1} = 27(7) + 6$$
$$x_{3,1} = 27(7^2) + 6(7) + 6$$
$$x_{4,1} = 27(7^3) + 6(7^2) + 6(7) + 6$$
$$x_{5,1} = 27(7^4) + 6(7^3) + 6(7^2) + 6(7) + 6$$
$$\vdots$$
$$x_{b,1} = 27(7^{b-1}) + 6(7^{b-2}) + \cdots + 6(7) + 6$$

$$y^r_{1,1} = 29$$
$$y^r_{2,1} = 29(7) - 6$$
$$y^r_{3,1} = 29(7^2) - 6(7) - 6$$
$$y^r_{4,1} = 29(7^3) - 6(7^2) - 6(7) - 6$$
$$y^r_{5,1} = 29(7^4) - 6(7^3) - 6(7^2) - 6(7) - 6$$
$$\vdots$$
$$y^r_{b,1} = 29(7^{b-1}) - 6(7^{b-2}) - \cdots - 6(7) - 6$$

Note that $x_{b,1}$ and $y^r_{b,1}$ are obtained by applying the mathematical induction on b, given as follows:

i. $x_{b,1}$: The base case follows since $x_{1,1} = 27$. Suppose the result holds for $k > 1$. We deduce that the result holds for $k + 1$.

$$\begin{aligned} x_{k+1,1} &= 7x_{k,1} + 6 \\ &= 7(27(7^{k-1}) + 6(7^{k-2}) + 6(7^{k-3}) + \cdots + 6(7) + 6) + 6 \\ &= 27(7^{(k+1)-1}) + 6(7^{(k+1)-2}) + 6(7^{(k+1)-3}) + \cdots + 6(7) + 6. \end{aligned}$$

Therefore, the result is true.

ii. $y^r_{b,1}$: The base case follows since $y^r_{1,1} = 29$. Suppose the result holds for $k > 1$. We now deduce that the result holds for $k + 1$.

$$y^r_{k+1,1} = 7y^r_{k,1} - 6$$

$$
\begin{aligned}
&= \quad 7\left(29\left(7^{k-1}\right) - 6\left(7^{k-2}\right) - 6\left(7^{k-3}\right) - \cdots - 6(7) - 6\right) - 6 \\
&= \quad 29\left(7^{(k+1)-1}\right) - 6\left(7^{(k+1)-2}\right) - 6\left(7^{(k+1)-3}\right) - \cdots - 6(7) - 6.
\end{aligned}
$$

Therefore, the result is also true.

Thus, for each i, we obtain the following equations:

$$
\begin{aligned}
x_{b,1} &= 27\left(7^{b-1}\right) + 6\left(7^{b-2}\right) + \cdots + 6(7) + 6 \\
x_{b,2} &= 12\left(7^{b-1}\right) + 12\left(7^{b-2}\right) + \cdots + 12(7) + 12 \\
x_{b,3} &= 3\left(7^{b-1}\right) + 24\left(7^{b-2}\right) + \cdots + 24(7) + 24 \\
x_{b,4} &= 27\left(7^{b-2}\right) + 6\left(7^{b-3}\right) + \cdots + 6(7) \\
x_{b,5} &= 12\left(7^{b-2}\right) + 12\left(7^{b-3}\right) + \cdots + 12(7) \\
x_{b,6} &= 3\left(7^{b-2}\right) + 24\left(7^{b-3}\right) + \cdots + 24(7) \\
x_{b,7} &= 27\left(7^{b-3}\right) + 6\left(7^{b-4}\right) + \cdots + 6\left(7^2\right) \\
x_{b,8} &= 12\left(7^{b-3}\right) + 12\left(7^{b-4}\right) + \cdots + 12\left(7^2\right) \\
x_{b,9} &= 3\left(7^{b-3}\right) + 24\left(7^{b-4}\right) + \cdots + 24\left(7^2\right) \\
x_{b,10} &= 27\left(7^{b-4}\right) + 6\left(7^{b-5}\right) + \cdots + 6\left(7^3\right) \\
x_{b,11} &= 12\left(7^{b-4}\right) + 12\left(7^{b-5}\right) + \cdots + 12\left(7^3\right) \\
x_{b,12} &= 3\left(7^{b-4}\right) + 24\left(7^{b-5}\right) + \cdots + 24\left(7^3\right) \\
&\vdots
\end{aligned}
$$

$$
\begin{aligned}
y_{b,1}^r &= 29\left(7^{b-1}\right) - 6\left(7^{b-2}\right) - \cdots - 6(7) - 6 \\
y_{b,2}^r &= 16\left(7^{b-1}\right) - 12\left(7^{b-2}\right) - \cdots - 12(7) - 12 \\
y_{b,3}^r &= 11\left(7^{b-1}\right) - 24\left(7^{b-2}\right) - \cdots - 24(7) - 24 \\
y_{b,4}^r &= 29\left(7^{b-2}\right) - 6\left(7^{b-3}\right) - \cdots - 6(7) \\
y_{b,5}^r &= 16\left(7^{b-2}\right) - 12\left(7^{b-3}\right) - \cdots - 12(7) \\
y_{b,6}^r &= 11\left(7^{b-2}\right) - 24\left(7^{b-3}\right) - \cdots - 24(7) \\
y_{b,7}^r &= 29\left(7^{b-3}\right) - 6\left(7^{b-4}\right) - \cdots - 6\left(7^2\right) \\
y_{b,8}^r &= 16\left(7^{b-3}\right) - 12\left(7^{b-4}\right) - \cdots - 12\left(7^2\right) \\
y_{b,9}^r &= 11\left(7^{b-3}\right) - 24\left(7^{b-4}\right) - \cdots - 24\left(7^2\right) \\
y_{b,10}^r &= 29\left(7^{b-4}\right) - 6\left(7^{b-5}\right) - \cdots - 6\left(7^3\right) \\
y_{b,11}^r &= 16\left(7^{b-4}\right) - 12\left(7^{b-5}\right) - \cdots - 12\left(7^3\right) \\
y_{b,12}^r &= 11\left(7^{b-4}\right) - 24\left(7^{b-5}\right) - \cdots - 24\left(7^3\right) \\
&\vdots
\end{aligned}
$$

By observation, it is clear that the above equations can be classified into three categories according to the value of i, that is, when $i = 3t - 2$, $i = 3t - 1$ and $i = 3t$, where $t \in \mathbb{Z}^+$. In order to obtain the general forms for $x_{b,i}$ and $y_{b,i}^r$ in each category, we apply the induction on i.

We first consider the case when $i = 3t - 2$. Let

$$
x_{b,i} = 27\left(7^{b-\left(\frac{1}{3}i+\frac{2}{3}\right)}\right) + 6\left(7^{b-\left(\frac{1}{3}i+\frac{2}{3}\right)-1}\right) + 6\left(7^{b-\left(\frac{1}{3}i+\frac{2}{3}\right)-2}\right) + \cdots + 6\left(7^{\left(\frac{1}{3}i+\frac{2}{3}\right)}\right) + 6\left(7^{\left(\frac{1}{3}i+\frac{2}{3}\right)-1}\right).
$$

For the base case, suppose $i = 1$. It is clear that the result is true. Assume that the result holds for $i = k > 1$, it can be seen that

$$
\begin{aligned}
x_{b,k+3} &= 7^{-1} \cdot x_{b,k} - 6\left(7^{\frac{k}{3}-\frac{1}{3}}\right) - 6\left(7^{\frac{k}{3}-\frac{4}{3}}\right) \\[6pt]
&= 27\left(7^{b-\left(\frac{1}{3}k+\frac{2}{3}\right)-1}\right) + 6\left(7^{b-\left(\frac{1}{3}k+\frac{2}{3}\right)-2}\right) + 6\left(7^{b-\left(\frac{1}{3}k+\frac{2}{3}\right)-3}\right) + \cdots + 6\left(7^{\left(\frac{1}{3}k+\frac{2}{3}\right)+1}\right) + \\
&\quad\ 6\left(7^{\left(\frac{1}{3}k+\frac{2}{3}\right)}\right) + 6\left(7^{\left(\frac{1}{3}k+\frac{2}{3}\right)-1}\right) + 6\left(7^{\left(\frac{1}{3}k+\frac{2}{3}\right)-2}\right) - 6\left(7^{\frac{k}{3}-\frac{1}{3}}\right) - 6\left(7^{\frac{k}{3}-\frac{4}{3}}\right) \\[6pt]
&= 27\left(7^{b-\left(\frac{1}{3}k+\frac{2}{3}\right)-1}\right) + 6\left(7^{b-\left(\frac{1}{3}k+\frac{2}{3}\right)-2}\right) + 6\left(7^{b-\left(\frac{1}{3}k+\frac{2}{3}\right)-3}\right) + \cdots + 6\left(7^{\left(\frac{1}{3}k+\frac{2}{3}\right)+1}\right) + \\
&\quad\ 6\left(7^{\left(\frac{1}{3}k+\frac{2}{3}\right)}\right) \\[6pt]
&= 27\left(7^{b-\left(\frac{1}{3}(k+3)+\frac{2}{3}\right)}\right) + 6\left(7^{b-\left(\frac{1}{3}(k+3)+\frac{2}{3}\right)-1}\right) + 6\left(7^{b-\left(\frac{1}{3}(k+3)+\frac{2}{3}\right)-2}\right) + \cdots + \\
&\quad\ 6\left(7^{\left(\frac{1}{3}(k+3)+\frac{2}{3}\right)}\right) + 6\left(7^{\left(\frac{1}{3}(k+3)+\frac{2}{3}\right)-1}\right).
\end{aligned}
$$

Therefore, the result is also true for $i = k + 3$. Hence, we have

$$x_{b,i} = \begin{array}{l} 27\left(7^{b-\left(\frac{1}{3}i+\frac{2}{3}\right)}\right) + 6\left(7^{b-\left(\frac{1}{3}i+\frac{2}{3}\right)-1}\right) + 6\left(7^{b-\left(\frac{1}{3}i+\frac{2}{3}\right)-2}\right) + \cdots + 6\left(7^{\left(\frac{1}{3}i+\frac{2}{3}\right)}\right) + \\ 6\left(7^{\left(\frac{1}{3}i+\frac{2}{3}\right)-1}\right). \end{array}$$

Next, we let

$$y_{b,i}^r = \begin{array}{l} 27\left(7^{b-\left(\frac{1}{3}i+\frac{2}{3}\right)}\right) - 6\left(7^{b-\left(\frac{1}{3}i+\frac{2}{3}\right)-1}\right) - 6\left(7^{b-\left(\frac{1}{3}i+\frac{2}{3}\right)-2}\right) - \cdots - 6\left(7^{\left(\frac{1}{3}i+\frac{2}{3}\right)}\right) - \\ 6\left(7^{\left(\frac{1}{3}i+\frac{2}{3}\right)-1}\right). \end{array}$$

For the base case, suppose $i = 1$. It is clear that the result is true. Assume that the result is also true for $i = k > 1$, it can be seen that

$$y_{b,k+3}^r = 7^{-1} \cdot y_{b,k}^r + 6\left(7^{\frac{k}{3}-\frac{1}{3}}\right) + 6\left(7^{\frac{k}{3}-\frac{4}{3}}\right)$$

$$= \begin{array}{l} 27\left(7^{b-\left(\frac{1}{3}k+\frac{2}{3}\right)-1}\right) - 6\left(7^{b-\left(\frac{1}{3}k+\frac{2}{3}\right)-2}\right) - 6\left(7^{b-\left(\frac{1}{3}k+\frac{2}{3}\right)-3}\right) - \cdots - 6\left(7^{\left(\frac{1}{3}k+\frac{2}{3}\right)+1}\right) - \\ 6\left(7^{\left(\frac{1}{3}k+\frac{2}{3}\right)}\right) - 6\left(7^{\left(\frac{1}{3}k+\frac{2}{3}\right)-1}\right) - 6\left(7^{\left(\frac{1}{3}k+\frac{2}{3}\right)-2}\right) + 6\left(7^{\frac{k}{3}-\frac{1}{3}}\right) + 6\left(7^{\frac{k}{3}-\frac{4}{3}}\right) \end{array}$$

$$= \begin{array}{l} 27\left(7^{b-\left(\frac{1}{3}k+\frac{2}{3}\right)-1}\right) - 6\left(7^{b-\left(\frac{1}{3}k+\frac{2}{3}\right)-2}\right) - 6\left(7^{b-\left(\frac{1}{3}k+\frac{2}{3}\right)-3}\right) - \cdots - 6\left(7^{\left(\frac{1}{3}k+\frac{2}{3}\right)+1}\right) - \\ 6\left(7^{\left(\frac{1}{3}k+\frac{2}{3}\right)}\right) \end{array}$$

$$= \begin{array}{l} 27\left(7^{b-\left(\frac{1}{3}(k+3)+\frac{2}{3}\right)}\right) - 6\left(7^{b-\left(\frac{1}{3}(k+3)+\frac{2}{3}\right)-1}\right) - 6\left(7^{b-\left(\frac{1}{3}(k+3)+\frac{2}{3}\right)-2}\right) - \cdots - \\ 6\left(7^{\left(\frac{1}{3}(k+3)+\frac{2}{3}\right)}\right) - 6\left(7^{\left(\frac{1}{3}(k+3)+\frac{2}{3}\right)-1}\right). \end{array}$$

Therefore, the result is also true for $i = k + 3$. Hence, we have

$$y_{b,i}^r = \begin{array}{l} 27\left(7^{b-\left(\frac{1}{3}i+\frac{2}{3}\right)}\right) - 6\left(7^{b-\left(\frac{1}{3}i+\frac{2}{3}\right)-1}\right) - 6\left(7^{b-\left(\frac{1}{3}i+\frac{2}{3}\right)-2}\right) - \cdots - 6\left(7^{\left(\frac{1}{3}i+\frac{2}{3}\right)}\right) - \\ 6\left(7^{\left(\frac{1}{3}i+\frac{2}{3}\right)-1}\right). \end{array}$$

By omitting the first term in both $x_{b,i}$ and $y_{b,i}^r$, they can then be simplified further by using the method of geometric progression. Let the common ratio $v = 7^{-1}$, the initial value $a = 7^{b-\left(\frac{1}{3}i+\frac{2}{3}\right)-1}$ and the number of terms $n = b - \frac{2}{3}i - \frac{1}{3}$, we obtain

$$x_{b,i} = 27\left(7^{b-\left(\frac{1}{3}i+\frac{2}{3}\right)}\right) + 6\left[\frac{7^{b-\left(\frac{1}{3}i+\frac{2}{3}\right)-1}\left(1 - 7^{-\left(b-\frac{2}{3}i-\frac{1}{3}\right)}\right)}{1 - 7^{-1}}\right]$$

$$= 7^{\frac{1}{3}i-\frac{1}{3}}\left(2^2 \cdot 7^{b-\frac{2}{3}i+\frac{2}{3}} - 1\right), \tag{4}$$

$$y_{b,i}^r = 29\left(7^{b-\left(\frac{1}{3}i+\frac{2}{3}\right)}\right) - 6\left[\frac{7^{b-\left(\frac{1}{3}i+\frac{2}{3}\right)-1}\left(1 - 7^{-\left(b-\frac{2}{3}i-\frac{1}{3}\right)}\right)}{1 - 7^{-1}}\right]$$

$$= 7^{\frac{1}{3}i-\frac{1}{3}}\left(2^2 \cdot 7^{b-\frac{2}{3}i+\frac{2}{3}} + 1\right). \tag{5}$$

Using a similar approach, the general forms for $x_{b,i}$ and $y_{b,i}^r$ for the two remaining cases can also be obtained, as follows:
When $i = 3t - 1$,

$$x_{b,i} = 7^{\frac{1}{3}i-\frac{2}{3}}\left(2 \cdot 7^{b-\frac{2}{3}i+\frac{4}{3}} - 2\right), \tag{6}$$

$$y_{b,i}^r = 7^{\frac{1}{3}i-\frac{2}{3}}\left(2 \cdot 7^{b-\frac{2}{3}i+\frac{4}{3}} + 2\right). \tag{7}$$

When $i = 3t$,

$$x_{b,i} = 7^{\frac{1}{3}i-1}\left(7^{b-\frac{2}{3}i+2} - 2^2\right), \tag{8}$$

$$y_{b,i}^r = 7^{\frac{1}{3}i-1}\left(7^{b-\frac{2}{3}i+2} + 2^2\right). \tag{9}$$

This completes the proof.   □

The Range of i

We now determine the range of i in $x_{b,i}$ and $y_{b,i}^r$ for each b. We also give the generators of solutions to the equation $x^2 + 16 \cdot 7^b = y^{2r}$ for some specific i.

Lemma 3. *Let $b, r, t \in \mathbb{Z}^+$. The ranges of i associated with each b in the equation $x^2 + 16 \cdot 7^b = y^{2r}$ are*

$$\begin{cases} 0 < i \le \dfrac{3}{2}(b+1), & \text{when } b \text{ is odd,} \\[2mm] 0 < i \le \dfrac{3}{2}b + 2, & \text{when } b \text{ is even.} \end{cases}$$

Proof. We consider three cases (when $i = 3t - 2$, $i = 3t - 1$ and $i = 3t$) to determine the ranges of i, corresponding to the three general forms we obtained in Theorem 1.

First, when $i = 3t - 2$, we have $x_{b,i} = 7^{\frac{1}{3}i - \frac{1}{3}}\left(2^2 \cdot 7^{b - \frac{2}{3}i + \frac{2}{3}} - 1\right)$, based on Eq. (4). Since $x_{b,i}$ is non-negative, we have

$$\begin{aligned} 4 \cdot 7^{b - \frac{2}{3}i + \frac{2}{3}} &> 1 \\ 7^{b - \frac{2}{3}i + \frac{2}{3}} &> \tfrac{1}{4} \\ \log 7^{b - \frac{2}{3}i + \frac{2}{3}} &> \log \tfrac{1}{4} \\ b - \tfrac{2}{3}i + \tfrac{2}{3} &> -0.7124 \\ i &< \tfrac{2}{3}b + 2.0686. \end{aligned}$$

This implies that $i < \left\lceil \frac{2}{3}b + 2.0686 \right\rceil$. Therefore, $i \le \frac{3}{2}(b+1)$ when b is odd and $i \le \frac{3}{2}b + 2$ when b is even.

By some appropriate modifications to the above method, we then obtain the same ranges for the two remaining cases, when b is odd and even. Therefore, the result follows.   □

By Lemma 3, we can see that $i \le \frac{3}{2}b + 2$ when b is even. When $0 < i < \frac{3}{2}b + 1$, we have Eqs. (4) to (9) as the generators of $x_{b,i}$ and $y_{b,i}^r$. When $i = \frac{3}{2}b + 2$ and $i = \frac{3}{2}b + 1$, the generators $x_{b,i}$ and $y_{b,i}^r$ attain different forms. The following two theorems discuss the generators in such cases.

Theorem 4. *Let $i = \frac{3}{2}b + 2$, b be an even number and r be any positive integer. Then, $x_{b,i} = 0$ and $y_{b,i}^r = 4 \cdot 7^{\frac{1}{2}b}$ are the generators of solutions to the equation $x^2 + 16 \cdot 7^b = y^{2r}$.*

Proof. Suppose $b = 2h$, where h is an integer. Then, we have $i = \frac{3}{2}(2h) + 2 = 3h + 2$. The set $\{i = 3h + 2 \mid h \in \mathbb{Z}\}$ is a subset of $\{i = 3t - 1 \mid t \in \mathbb{Z}\}$. Hence, we consider Eq. (6) in Theorem 1, that is

$$x_{b,i} = 7^{\frac{1}{3}i - \frac{2}{3}}\left(2 \cdot 7^{b - \frac{2}{3}i + \frac{4}{3}} - 2\right).$$

By substituting $i = \frac{3}{2}b + 2$, we obtain

$$x_{b,i} = 7^{\frac{1}{3}\left(\frac{3}{2}b + 2\right) - \frac{2}{3}}\left(2 \cdot 7^{b - \frac{2}{3}\left(\frac{3}{2}b + 2\right) + \frac{4}{3}} - 2\right) = 7^{\frac{1}{2}b}(2 - 2).$$

Thus,

$$x_{b,i} = 0.$$

Now, we consider Eq. (7) in Theorem 1, that is

$$y_{b,i}^r = 7^{\frac{1}{3}i - \frac{2}{3}}\left(2 \cdot 7^{b - \frac{2}{3}i + \frac{4}{3}} + 2\right).$$

By substituting $i = \frac{3}{2}b + 2$, we obtain

$$y_{b,i}^r = 7^{\frac{1}{3}\left(\frac{3}{2}b + 2\right) - \frac{2}{3}}\left(2 \cdot 7^{b - \frac{2}{3}\left(\frac{3}{2}b + 2\right) + \frac{4}{3}} + 2\right) = 4 \cdot 7^{\frac{1}{2}b}.$$

Thus, $x_{b,i} = 0$ and $y_{b,i}^r = 4 \cdot 7^{\frac{1}{2}b}$ are the generators of solutions that satisfy the equation $x^2 + 16 \cdot 7^b =$

y^{2r}, when $i = \frac{3}{2}b + 2$, b is an even number and r is any positive integer.   □

Theorem 5. *Let $i = \frac{3}{2}b + 1$, b be an even number and r be any positive integer. Then, $x_{b,i} = 3 \cdot 7^{\frac{1}{2}b}$ and $y_{b,i}^r = 5 \cdot 7^{\frac{1}{2}b}$ are the generators of solutions to the equation $x^2 + 16 \cdot 7^b = y^{2r}$.*

Proof. By using a similar approach as in the proof of Theorem 4.   □

Conclusions

Our results show that there exist three sets of generators of solutions to the equation $x^2 + 16 \cdot 7^b = y^{2r}$, according to the values of i. These generators are all given in Eqs. (4) to (9) in Theorem 1.

Since the value of i is associated with the value of b, we also determine the range of i based on the parity of b. We proved that when b is an odd number, we have $i \in \{1, 2, \ldots, \frac{3}{2}(b + 1)\}$. On the other hand, we obtain $i \in \{1, 2, \ldots, \frac{3}{2}b + 2\}$ when b is an even number.

Note that when b is an even number, the generators $(x_{b,i}, y_{b,i}^r)$ of solutions in Eqs. (4) to (9) can only be applied when the value of $i \leq \frac{3}{2}b$. Hence, two other sets of generators are given specifically for the cases when $i = \frac{3}{2}b + 2$ and $i = \frac{3}{2}b + 1$. The two sets of generators are shown in Theorems 4 and 5, and they each has a simpler form than the others. It is also clear that these results can be verified further by using the sequences (when b is even) provided in the proof of Theorem 1.

Conflicts of Interest

The author(s) declare(s) that there is no conflict of interest regarding the publication of this paper.

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