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Optimal Repair Rate for a Repairable Machine with Nonlinear State Equation

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Abstract This work deals with two models of a single machine subject to breakdowns. The first model is a binary-state model while the second one is a multi-state model. Breakdowns are followed by repairs. The systems under study are of the tracking type. We use nonlinear model predictive control to determine the optimal repair rates that keep the variables as close as possible to their targets. Numerical illustrations with sensitivity analysis are used to assess the effect of the system parameters on the optimal solution.

Keywords: Machine breakdown, minor breakdown, major breakdown, repair, model predictive control.

1. Introduction

Machine breakdown is an important issue that companies face. Machine breakdown causes business interruption which may result in considerable losses. For example, the air conditioning equipment of a resort hotel that breaks down during a summer holiday weekend, the sudden failure of a critical part in an airplane, the crash of a specialized molding machine during peak production time for a candy manufacturer, a medical device used in a hospital on which the life of a patient may depend. Examples are endless and results could be disastrous.

Breakdowns can be of different types. A minor breakdown may be repaired steadily whereas major breakdowns may keep the machine away for a long time. Catastrophes can be avoided by diagnosing the right breakdown and employing the right repair.

Following breakdowns, businesses need to repair the equipment as quickly as possible, which may result in additional costs to accelerate repairs. In all cases, breakdowns and repairs lead to a lost income and potentially expensive repairs or replacement of equipment.

Machine breakdown and replacement policies have been extensively studied by researchers under different angles. One of the earliest works is the book of Barlow and Proschan [1]. Many surveys of the literature are available e.g., Valdez-Florez and Feldman [9], Dekker [2], Rausand and Høyland [6], Marquez and Heguedas [4].

Among the characteristics of interest in a replacement policy are the objective function, the time of replacement, the type of replacement, and the failure time process, see Popova and Popova [5]. The solution techniques are also varied and include mathematical programming (integer programming, dynamic programming, chance programming, geometric programming), heuristics, and meta-heuristics (genetic algorithms, simulated annealing, tabu search, particle swarm optimization, ant colony optimization, honey bee mating optimization, artificial bee colony, harmony search, variable neighborhood search, variable neighborhood search descent), see Soltani [8] for a comprehensive review.

Our starting point is a problem that was introduced by Hartl [3]. Consider a machine that is prone to failure. The process is Markovian. At any instant of time, the machine can be either operable or not. Following breakdown, the machine undergoes repair. Given the state of the system, it is desired to obtain

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the optimal repair rate.

The model just described is known in the literature as a binary-state model. A more general model is the multi-state model where the number of states is greater than 2. We first study in this paper the binary-state model then generalize the results to a three-state model.

The considered models are dynamic and therefore an optimal control technique, model predictive control, seems appropriate. Model predictive control (MPC) is an advanced method of process control that is used to control a dynamic process. It has been in use in the process industries in chemical plants, oil refineries, power systems, and power electronics. The way MPC works is that it optimizes the current timeslot, while keeping future timeslots into account. This is achieved by optimizing a finite time horizon, but only implementing the current timeslot and then optimizing again, repeatedly. When the state equation is nonlinear, the method is called nonlinear model predictive control (NMPC).
Following this introduction, the binary-state model is studied in Section 2 and the three-state model considered in Section 3. Both sections contain illustrative examples. The paper is concluded in Section 4.

2 Binary-State Model

As mentioned in the Introduction section, we begin by studying a binary state model. We first introduce the notation and build the model then, as it turns out to be nonlinear, we solve it using NMPC.

2.1 Model formulation

Let $H > 0$ denote the length of the planning horizon and consider a machine that is subject at any time t to breakdowns at a rate $v(t)$. Denote by 0 and 1 the state of the machine when it is operable and not operable, respectively. Following a breakdown, the machine is brought back to the operable state at a rate $u(t)$. In this setting, the respective probabilities $p_0(t)$ and $p_1(t)$ of states 0 and 1 are the state variables and the repair rate $u(t)$ is the control variable. The variables $v(t)$ is an exogenous variable. The diagram associated with this system is depicted in Figure 1. The dynamics of this system can be represented by the following equations:

$$\begin{cases} \frac{d}{dt}p_0(t) & = -v(t)p_0(t) + u(t)p_1(t), \\ \frac{d}{dt}p_1(t) & = v(t)p_0(t) - u(t)p_1(t), \end{cases} \tag{2,1}$$

where the initial conditions are supposed to be known. Let $x(t) = p_0(t)$. Since $p_0(t) + p_1(t) = 1$, we get the system state equation as follows:

$$\frac{d}{dt}x(t) = -v(t)x(t) + u(t)[1 - x(t)].$$

We are assuming a system of the tracking-type. We let $\hat{x}(t)$ denote the target probability that the machine is operable at time t. The corresponding target

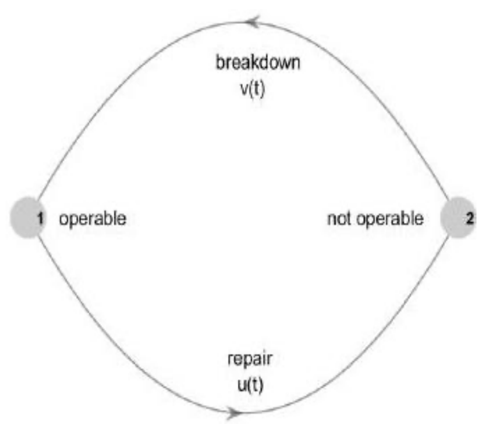


Figure 1: Binary-state model.

Repair rate at time t is denoted by $\hat{u}(t)$. Since the pair $(\hat{x}(t), \hat{u}(t))$ must be admissible, we have

$$\hat{u}(t) = \frac{1}{1-\hat{x}(t)}\left[v(t)\hat{x}(t) + \frac{d}{dt}\hat{x}(t)\right]. \tag{2.2}$$

Given targets for the variables involved, the objective is to obtain the optimal repair rate so that each variable converges to its target. In order to achieve this goal, we let $t_0 \in [0, H], T > 0$ with $T \ll H$ and introduce the objective function to minimize:

$$J(t_0, x, u) = \frac{1}{2}\int_{t_0}^{t_0+T} \{p\Delta x(t)^2 + q\Delta u(t)^2\}dt + \frac{c}{2}\Delta x(t_0 + T)^2. \tag{2.3}$$

The interval $[t_0, t_0 + T]$ is called prediction interval. The shift operator Δ measures the gap between a variable and its target $\Delta f(t) = f(t) - \hat{f}(t)$ and p, q are penalties incurred when a variable deviates from its goal. Also, c is the final state penalty. The objective function (2.3) is commonly used in engineering and management science, see for example Sethi and Thompson [7].

The state equation is rewritten in terms of the Δ operator as follows:

$$\frac{d}{dt}\Delta x(t) = -[v(t) + \hat{u}(t)]\Delta x(t) + [1 - x(t)]\Delta u(t). \tag{2.4}$$

The problem then is to minimize the objective function (2.3) where the control variable is the repair rate $u(t)$, subject to the nonlinear state equation (2.4).

2.2 Model solution

The prediction interval $[t_0, t_0 + T]$ is divided into m subintervals of equal length $h = T/m$ and the trapezoid formula of numerical analysis is used to calculate the integral in the objective function (2.3):

$$J(t_0, x, u) = \frac{h}{2}\left[F(t_0) + 2\sum_{i=1}^{m-1} F(t_0 + ih) + F(t_0 + mh)\right] + \frac{c}{2}\Delta x(t_0 + mh)^2, \tag{2.5}$$

where

$$F(t) = \frac{1}{2}[p\Delta x(t)^2 + q\Delta u(t)^2].$$

For simplicity we will write t instead of t_0. The terms $F(t + ih)$ are calculated by combining Taylor's expansion with the state equation (2.4). Following some lengthy calculation, we rewrite the objective function (2.5) in the following matrix-vector notation:

$$J(t, x, u) = a_0(t) + A_1(t)^{\top}U(t) + U(t)^{\top}A_2(t)U(t), \tag{2.6}$$

where $a_0(t)$ is independent of the control variable $u(t)$, the vectors $U(t)$ and $A_1(t)$ are of dimension $m \times 1$, the matrix $A_2(t)$ is of dimension $m \times m$,

$$U(t) = \begin{bmatrix} \Delta u(t) \\ \Delta u(t+h) \\ \vdots \\ \Delta u(t + (m-1)h) \end{bmatrix}, A_1(t) = \begin{bmatrix} a_1(t) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, A_2(t) = \begin{bmatrix} a_2(t) & 0 & \cdots & 0 \\ 0 & \frac{hq}{2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{hq}{2} \end{bmatrix},$$

with

$$\begin{aligned} a_0(t) = {} & \frac{hp}{4}\Delta x(t)^2 + \frac{hp}{2}\{(m-1) - 2\alpha h[v(t) + \hat{u}(t)] + \beta h^2[v(t) + \hat{u}(t)]\}\Delta x(t)^2 \\ & + \left(\frac{hp}{4} + \frac{c}{2}\right)\{1 - mh[v(t) + \hat{u}(t)]\}^2\Delta x(t)^2 \\ a_1(t) = {} & \left\{\alpha h^2 p + hm\left(\frac{hp}{2} + c\right)\right. \\ & \left. - \left[\beta h^3 p + h^2 m^2\left(\frac{hp}{2} + c\right)\right][v(t) + \hat{u}(t)]\right\}[1 - x(t)]\Delta x(t) \end{aligned}$$

$$a_2(t) = \frac{hq}{4} + h^2\left[\frac{\beta hp}{2} + m^2\left(\frac{hp}{4} + \frac{c}{2}\right)\right][1 - x(t)]^2.$$

and

$$\alpha = \frac{m(m-1)}{2},\ \beta = \frac{m(m-1)(2m-1)}{6}.$$

The objective function (2.6) is unimodal and the necessary and sufficient optimality condition yields

$$U^*(t) = -\tfrac{1}{2}A_2(t)^{-1}A_1(t), \tag{2.7}$$

from which we obtain the optimal repair rate:

$$\Delta u^*(t) = -\frac{a_1(t)}{2a_2(t)}. \tag{2.8}$$

However, $\Delta u^*(t)$ is found in terms of $\Delta x^*(t)$. Inserting this expression of $\Delta u^*(t)$ in the state equation (2.4) yields a nonlinear differential equation that is solved numerically. Finally, the solution of the differential equation is substituted back in (2.8) to obtain the optimal intensity of repair. Note that the optimal objective function value can be found by substituting (2.7) into (2.6) to get

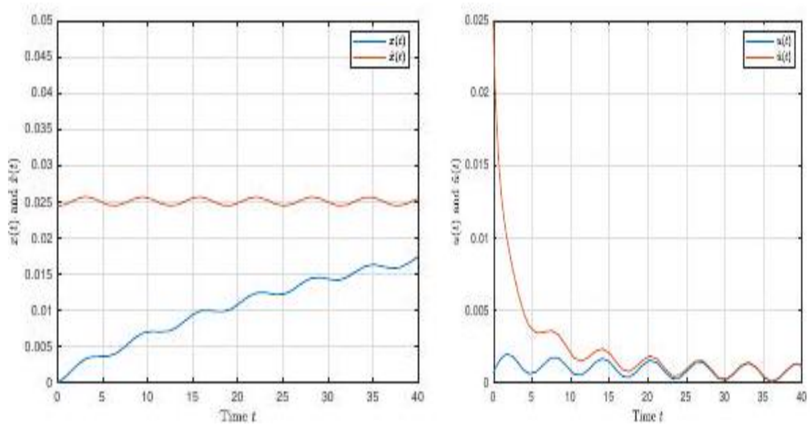
$$J(t, x^*, u^*) = a_0(t) - \frac{a_1(t)^2}{4a_2(t)}. \tag{2.9}$$

2.3 Numerical example

We present an illustrative example to validate the theoretical results. Assume the following system parameters $T = 40, t_0 = 0, m = 80, p = 1,\ q = 10, c = 50$ and let the repair rate be $v(t) = \frac{1}{1+t}$ and the target probability that the machine is operable $\hat{x}(t) = \frac{1}{40+\cos(t)}$. Using (2.2) we find the target repair rate

$$\hat{u}(t) = \frac{1}{(39+\cos(t))(40+\cos(t))}\left[\frac{1}{1+t} + \frac{\sin(t)}{40+\cos(t)}\right].$$

Figure 2 shows the convergence of the optimal state and control variables towards their respective goals.



(a) Optimal state variable $x^*(t)$ (b) Optimal control variable $u^*(t)$

Figure 2: Optimal solution of the binary-state model.

Sensitivity analysis. Once the optimal state and control variables have been found, it is possible to do a sensitivity analysis to assess the effect of the system parameters on the optimal objective function value. For example, if we are interested in the effect of the prediction horizon length, we can calculate J^* for different values of T, keeping all other variables fixed. The result is displayed in Figure 3. It shows that the shorter the prediction horizon, the lower the optimal objective function value.

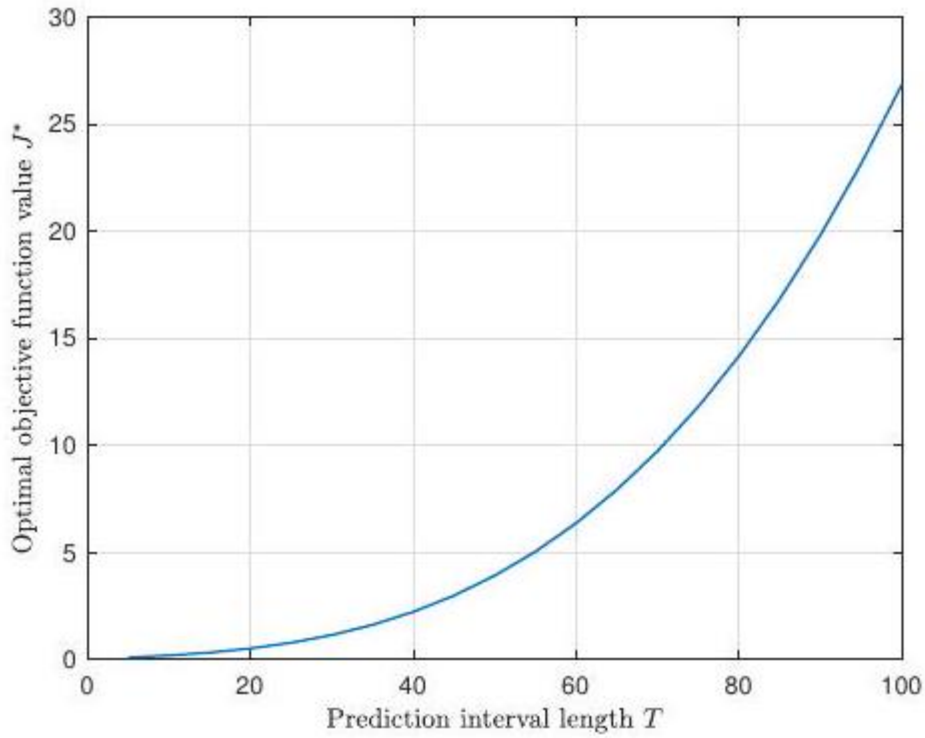


Figure 3: Variations of J^* as a function of T.

3 Three-State Model

As mentioned in the Introduction section, we now generalize the results of the binary-state model to a multi-state model by considering a three-state model. As in the previous section, the model is built and then solved using NMPC.

3.1 Model formulation

Consider a machine that is subject at any time t to two types of breakdowns: a minor breakdown that happens at a rate $v_1(t)$ and a major breakdown that happens at a rate $v_2(t)$. Denote by 0, 1 and 2 the state of the machine when it is operable, after a minor breakdown, and after a major breakdown, respectively. Following a minor breakdown, the machine is brought back to the operable state at a rate $u_1(t)$, and following a major breakdown, the machine is brought back to the operable state at a rate $u_2(t)$. In this setting, $p_0(t), p_1(t), p_2(t)$ are the state variables and $u_1(t), u_2(t)$ are the control variables. The variables $v_1(t), v_2(t)$ are exogenous variables. The diagram associated with this system is depicted in Figure 4. The dynamics of this

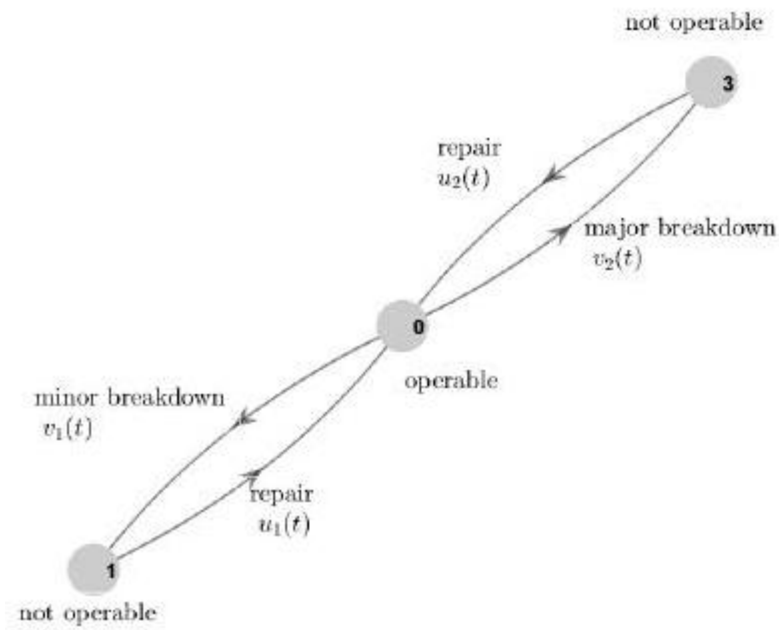


Figure 4: Three-state model.

system can be represented by the following equations:

$$\begin{cases} \frac{dp_0(t)}{dt} = -[v_1(t) + v_2(t)]p_0(t) + u_1(t)p_1(t) + u_2(t)p_2(t) \\ \frac{dp_1(t)}{dt} = -u_1(t)p_1(t) + v_1(t)p_0(t) \\ \frac{dp_2(t)}{dt} = -u_2(t)p_2(t) + v_2(t)p_0(t) \end{cases} \tag{3.1}$$

where the initial conditions are supposed to be known. Since $p_0(t) + p_1(t) + p_2(t) = 1$, we let $x_1(t) = p_0(t), x_2(t) = p_1(t)$ and reduce the above threeequation differential system to the following two-equation differential system:

$$\frac{dx_1(t)}{dt} = -[v_1(t) + v_2(t)]x_1(t) + u_1(t)x_2(t) + u_2(t)[1 - x_1(t) - x_2(t)]$$
$$\frac{dx_2(t)}{dt} = -u_1(t)x_2(t) + v_1(t)x_1(t)$$

As with the previous model, we are assuming a system of the tracking type. We let $\hat{x}_i(t)(i = 1,2)$ the target corresponding to the state variable $x_i(t)(i = 1,2)$, and $\hat{u}_i(t)(i = 1,2)$ the target repair rate corresponding to the control variable $\hat{u}_i(t)(i = 1,2)$. Since the pairs $(\hat{x}_i(t), \hat{u}_i(t))(i = 1,2)$ must be admissible, we have

$$\hat{u}_1(t) = \frac{1}{\hat{x}_2(t)}\left[v_1(t)\hat{x}_1(t) - \frac{d}{dt}\hat{x}_2(t)\right], \tag{3.2}$$

$$\hat{u}_2(t) = \frac{1}{1 - \hat{x}_1(t) - \hat{x}_2(t)}\left[v_2(t)\hat{x}_1(t) + \frac{d}{dt}\hat{x}_1(t) + \frac{d}{dt}\hat{x}_2(t)\right]. \tag{3.3}$$

Introducing the shift operator Δ, penalties p_i, q_i, and c_i, the problem is to determine the optimal repair rates $u_i(t)(i = 1,2)$ that minimize the performance index

$$J(t_0, x, u) = \frac{1}{2}\int_{t_0}^{t_0+T} \{p_1\Delta x_1(t)^2 + p_2\Delta x_2(t)^2 + q_1\Delta u_1(t)^2 + q_2\Delta u_2(t)^2\}dt + \frac{c_1}{2}\Delta x_1(t_0 + T)^2 + \frac{c_2}{2}\Delta x_2(t_0 + T)^2 \tag{3.4}$$

Subject to the nonlinear state equations

$$\frac{d}{dt}\Delta x_1(t) = -[v_1(t)+v_2(t)+\hat{u}_2(t)]\Delta x_1(t)+[\hat{u}_1(t)-\hat{u}_2(t)]\Delta x_2(t), \\ +x_2(t)\Delta u_1(t)+[1-x_1(t)-x_2(t)]\Delta u_2(t) \quad (3.5)$$

$$\frac{d}{dt}\Delta x_2(t) = v_1(t)\Delta x_1(t)-\hat{u}_1(t)\Delta x_2(t)-x_2(t)\Delta u_1(t). \quad (3.6)$$

3.2 Model solution

Using the trapezoid formula, Taylor's expansion, and proceeding as in the previous section, we rewrite the objective function (3.4) in the following matrix-vector notation:

$$\begin{aligned} J(t,x,u) = {} & a_0(t)+A_1(t)^\top U_1(t)+U_1(t)^\top A_2(t)U_1(t) \\ & +B_1(t)^\top U_2(t)+U_2(t)^\top B_2(t)U_2(t) \\ & +U_1(t)^\top C(t)U_2(t), \end{aligned} \quad (3.7)$$

Where $a_0(t)$ is independent of the control variables $u(t), u_2(t)$, the vectors $U_1(t), U_2(t)$ and $A_1(t), B_1(t)$ are of dimension $m\times 1$, the matrices $A_2(t), B_2(t), C(t)$ are of dimension $m\times m$:

$$U_i(t)=\begin{bmatrix}\Delta u_i(t)\\ \Delta u_i(t+h)\\ \vdots \\ \Delta u_i(t+(m-1)h)\end{bmatrix},\ A_i(t)=\begin{bmatrix}a_i(t)\\0\\ \vdots\\0\end{bmatrix},$$

$$B_i(t)=\begin{bmatrix} b_i(t) & 0 & \cdots & 0\\ 0 & \frac{hq_i}{2} & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{hq_i}{2}\end{bmatrix},\ C(t)=\begin{bmatrix} c(t) & 0 & \cdots & 0\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & 0\end{bmatrix},$$

with

$$\begin{aligned} a_0(t) = {} & \frac{h}{4}[p_1\Delta x_1(t)^2+p_2\Delta x_2(t)^2] \\ & +\frac{hp_1}{2}(\{m-1-2\alpha h[v_1(t)+v_2(t)+\hat{u}_2(t)]+\beta h^2[v_1(t)+v_2(t)+\hat{u}_2(t)]^2\}\Delta x_1(t)^2 \\ & +\beta h^2[\hat{u}_1(t)-\hat{u}_2(t)]^2\Delta x_2(t)^2 \\ & +2h\{\alpha-\beta h[v_1(t)+v_2(t)+\hat{u}_2(t)]\}[\hat{u}_1(t)-\hat{u}_2(t)]\Delta x_1(t)\Delta x_2(t)) \\ & +\frac{hp_2}{2}(\beta h^2 v_1(t)^2\Delta x_1(t)^2+[m-1-2\alpha h\hat{u}_1(t)+\beta h^2\hat{u}_1(t)^2]\Delta x_2(t)^2 \\ & +2hv_1(t)[\alpha-\beta h\hat{u}_1(t)]\Delta x_1(t)\Delta x_2(t)) \\ & +\left(\frac{hp_1}{4}+\frac{c_1}{2}\right)(\{1-hm[v_1(t)+v_2(t)+\hat{u}_2(t)]\}\Delta x_1(t)+hm[\hat{u}_1(t)-\hat{u}_2(t)]\Delta x_2(t))^2 \\ & +\left(\frac{hp_2}{4}+\frac{c_2}{2}\right)(hmv_1(t)\Delta x_1(t)+[1-hm\hat{u}_1(t)]\Delta x_2(t))^2 \\ a_1(t) = {} & \{p_1h^2\{\alpha-\beta h[v_1(t)+v_2(t)+\hat{u}_2(t)]\}-\beta p_2h^3v_1(t) \\ & +2hm\left(\frac{hp_1}{4}+\frac{c_1}{2}\right)\{1-hm[v_1(t)+v_2(t)+\hat{u}_2(t)]\}-2h^2m^2\left(\frac{hp_2}{4}+\frac{c_2}{2}\right)v_1(t)\Big\}\Delta x_1(t) \\ & +\Big\{\beta p_1h^3[\hat{u}_1(t)-\hat{u}_2(t)]+[\alpha-\beta h\hat{u}_1(t)]+2h^2m^2\left(\frac{hp_1}{4}+\frac{c_1}{2}\right)[\hat{u}_1(t)-\hat{u}_2(t)] \\ & -2hm\left(\frac{hp_2}{4}+\frac{c_2}{2}\right)[1-hm\hat{u}_1(t)]\Big\}\Delta x_2(t), \\ a_2(t) = {} & \frac{hq_1}{4}+\left[\left(\frac{p_1}{2}+\frac{p_2}{2}\right)\beta h^3+\left(\frac{hp_1}{4}+\frac{c_1}{2}\right)h^2m^2+\left(\frac{hp_2}{4}+\frac{c_2}{2}\right)h^2m^2\right]x_2(t)^2, \\ b_1(t) = {} & (p_1h^2\{\alpha-\beta h[v_1(t)+v_2(t)+\hat{u}_2(t)]\} \\ & +2hm\left(\frac{hp_1}{4}+\frac{c_1}{2}\right)\{1-hm[v_1(t)+v_2(t)+\hat{u}_2(t)]\}\Big)\Delta x_1(t) \\ & +\left(\beta p_1h^3[\hat{u}_1(t)-\hat{u}_2(t)]+2h^2m^2\left(\frac{hp_1}{4}+\frac{c_1}{2}\right)[\hat{u}_1(t)-\hat{u}_2(t)]\right)\Delta x_2(t), \\ c(t) = {} & \left\{\beta p_1h^3+\left(\frac{hp_1}{4}+\frac{c_1}{2}\right)2h^2m^2\right\}[1-x_1(t)-x_2(t)]x_2(t). \\ b_2(t) = {} & \frac{hq_2}{4}+\left[\frac{\beta p_1h^3}{2}+\left(\frac{hp_1}{4}+\frac{c_1}{2}\right)h^2m^2\right][1-x_1(t)-x_2(t)]^2, \end{aligned}$$

The objective function (3.7) is concave upward and the necessary and sufficient optimality conditions yield

$$U_1^*(t) = [4A_2(t)B_2(t) - C(t)^2]^{-1}[C(t)B_1(t) - 2B_2(t)A_1(t)], \tag{3.8}$$
$$U_2^*(t) = [4A_2(t)B_2(t) - C(t)^2]^{-1}[C(t)A_1(t) - 2A_2(t)B_1(t)], \tag{3.9}$$

from which we obtain the optimal repair rates:

$$\Delta u_1^*(t) = \frac{2a_1(t)b_2(t) - b_1(t)c(t)}{c(t)^2 - 4a_2(t)b_2(t)} \tag{3.10}$$
$$\Delta u_2^*(t) = \frac{2a_2(t)b_1(t) - a_1(t)c(t)}{c(t)^2 - 4a_2(t)b_2(t)} \tag{3.11}$$

The repair rates $\Delta u_1^*(t)$ and $\Delta u_2^*(t)$ are found in terms of $\Delta x_1^*(t)$ and $\Delta x_2^*(t)$. Inserting these expressions of $\Delta u_1^*(t)$ and $\Delta u_2^*(t)$ in the state equations (3.5)-(3.6) yields a nonlinear differential system that is solved numerically. Finally, the solutions of the differential system are substituted back in (3.10)-(3.11) to obtain the optimal intensities of repair. The optimal objective function value can be found by substituting (3.8)-(3.9) into (3.7) to get the following expression from which the time parameter t has been dropped not to encumber the equation:

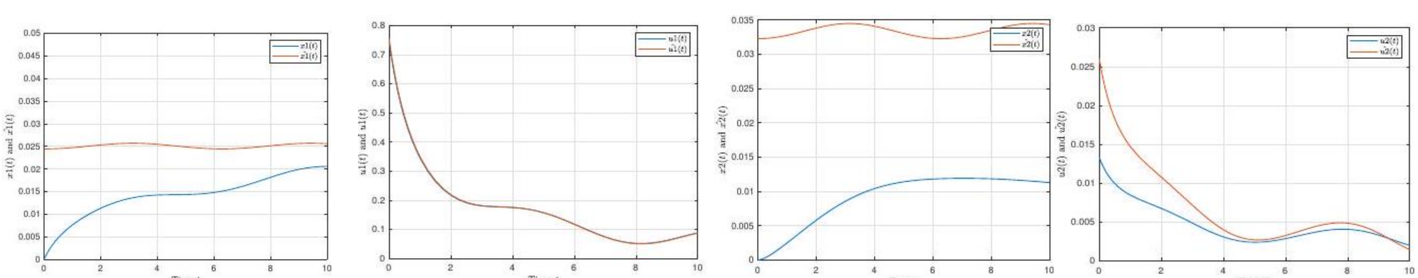
$$\begin{aligned} J(t, x^*, u^*) = \quad & a_0 + \frac{1}{(c^2 - 4a_2)^2}[(2a_1b_2 - b_1c)(a_1c^2 - 2a_1a_2b_2 - a_2b_1c) \\ & +(2a_2b_1 - a_1c)(b_1c^2 - 2a_2b_1b_2 - a_1b_2c) \\ & +c(2a_1b_2 - b_1c)(2a_2b_1 - a_1c)] \end{aligned} \tag{3.12}$$

3.3 Numerical example

We present an example to illustrate the theoretical results obtained in this section. Assume the following system parameters$T = 10, t_0 = 0, m = 50,\ p_1 = 2, p_2 = 1, q_1 = 10, q_2 = 6, c_1 = 30, c_2 = 10$, and let repair rates $v_i(t) = \frac{1}{1+t}, (i = 1,2)$ and the target probabilities $\hat{x}_1(t) = \frac{1}{40+\cos(t)}, \hat{x}_2(t) = \frac{1}{30+\cos(t)}$. Using (3.2)-(3.3) we find the target repair rates

$$\begin{aligned} \hat{u}_1(t) = \quad & \frac{30 + \cos(t)}{(1+t)[40 + \cos(t)]} - \frac{\sin(t)}{30 + \cos(t)}, \\ \hat{u}_2(t) = \quad & \frac{1}{[40 + \cos(t)][30 + \cos(t)] - 1}\left[\frac{30 + \cos(t)}{1+t} + \frac{\sin(t)[30 + \cos(t)]}{40 + \cos(t)} \right. \\ & \left. + \frac{\sin(t)\,[40 + \cos(t)]}{30 + \cos(t)}\right]. \end{aligned}$$

Figure 5: it shows the convergence of the optimal state and control variables towards their respective goals.



(a) Optimal state variable $x_1^*(t)$ (b) Optimal control variable $u_1^*(t)$ (c) Optimal state variable $x_2^*(t)$ (d) Optimal control variable $u_2^*(t)$

Figure 5: Optimal solution of the multi-state model.

4 Conclusion

Nonlinear model predictive control is a powerful control technique that can be employed effectively to optimize the performance of a process. In this paper, NMPC has been used to determine the optimal repair rate(s) of a machine subject to failures. The expressions involved in the three-state Model was more complicated than the expressions involved in the binary model. It would be interesting to generalize the results obtained in this work to an n-state model.

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