Journal of Fundamental Sciences Vol. 7, No. 1 (2011), 72-77.

ISSN 1823-626X

Journal of Fundamental Sciences

available online at http://jfs.ibnusina.utm.my

Statistical convergence of *n*-sequences and η-dual of some classical sets of *n*-sequences

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Received 20 March 2011, Revised 5 April 2011, Accepted 20 May 2011, Available online 28 June 2011

ABSTRACT

In this paper we introduce the notion of *n*-sequence and extend the notion of statistical convergence to *n*-sequences. Further we define the notion of $\eta -$ dual as a generalization of Köthe-Toeplitz dual for subsets of *n*-sequence spaces and compute $\eta -$ duals of some classical sets of *n*-sequences.

| *n*-sequence | statistical convergence | completeness | Köthe-Toeplitz dual |

http://dx.doi.org/10.11113/mjfas.v7n1.236

1. INTRODUCTION

Pringsheim gave the definition of the convergence for double sequences in 1900. Since then, this concept has been studied by many authors, see for instances [7, 14, 21]. The notion of the statistical convergence was first independently introduced by Fast [4] in 1951 and Schoenberg [17] in 1959. Later on it was further investigated from a sequence space point of view and linked with summability theory by Fridy [5], Salat [18] and many others. In [12] and [13] the above concept is extended to double sequences by using the idea of a two dimensional analogue of natural density

It is a fundamental principle of functional analysis that investigations of spaces are often combined with those of dual spaces. The notion of duals of sequence spaces was introduced by Köthe and Toeplitz [9]. Later on it was studied by Maddox [11], Lascarides [10], Bektaş, Et and Çolak [2], Chandra and Tripathy [3], Sarma [19], Dutta [1] and many others.

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2. DEFINITIONS AND PRELIMINARIES

Now we introduce some basic notions and examples related to the subject of this paper.

Definition 2.1: Let $n(\geq 2)$ be an integer. A function $x : N\times ...\times N(n-\text{factors})\to R(C)$ is called a real (complex) *n*-sequence, where *N*, *R* and *C* denote the sets of natural numbers, real numbers and complex numbers respectively.

Definition 2.2: An *n*-sequence $\left(x_{k_1...k_n}\right)$ is said to be convergent to L in Pringsheim's sense if for every $\varepsilon > 0$, there exists $M(\varepsilon)\in N$ such that

$$\left|x_{k_1...k_n} - L\right| < \varepsilon \text{ whenever } k_i \geq M, i = 1,...,n.$$

Example 2.1: Consider the 4-sequence $\left(x_{k_1k_2k_3k_4}\right)$, where

$$x_{k_1k_2k_3k_4} = \begin{cases} k_2k_3k_4, k_1 = 2 \\ k_1k_3k_4, k_2 = 4 \\ k_1k_2k_4, k_3 = 6 \\ k_1k_2k_3, k_4 = 8 \\ 10, \text{otherwise.} \end{cases}$$

Then $\left(x_{k_1k_2k_3k_4}\right)$ converges to 10 in Pringsheim's sense.

Definition 2.3: An *n*-sequence $\left(x_{k_1 \dots k_n}\right)$ is said to be a Cauchy sequence if for every $\varepsilon > 0$, there exists $M\left(\varepsilon\right) \in N$ such that

$$\left|x_{k_1 \dots k_n} - x_{m_1 \dots m_n}\right| < \varepsilon, \text{ whenever } k_i \geq m_i \geq M, i = 1, \dots, n.$$

Definition 2.4: An *n*-sequence $\left(x_{k_1 \dots k_n}\right)$ is said to be bounded if there exists $U > 0$ such that $\left|x_{k_1 \dots k_n}\right| < U$ for all $k_i, i = 1, \dots, n$.

We denote the set of all bounded *n*-sequences by ${}_n\ell_\infty$. It is easy to show that ${}_n\ell_\infty$ is a normed space, normed by

$$\|x\|_{(\infty,n)} = \sup_{k_1,\dots,k_n} \left|x_{k_1 \dots k_n}\right|.$$

A subset K of $N \times \dots \times N\left(n - \text{factors}\right)$ is said to have natural density $\delta_n\left(K\right)$ if

$$\delta_n\left(K\right) = \lim_{k_1,\dots,k_n \to \infty} \frac{\left|K\left(k_1,\dots,k_n\right)\right|}{k_1 \dots k_n} \text{ exists.}$$

Here $\left|K\left(k_1,\dots,k_n\right)\right|$ denotes the numbers of $\left(l_1,\dots,l_n\right)$ in K such that $l_i \leq k_i, i = 1, \dots, n$.

Example 2.2: Consider the set

$$K = \left\{\left(l_1^3, l_2^3, l_3^3, l_4^3, l_5^3\right) : l_1, l_2, l_3, l_4, l_5 \in N\right\} \subseteq N \times N \times N \times N \times N.$$

Then

$$\delta_5(K) = \lim_{k_1,k_2,k_3,k_4,k_5 \to \infty} \frac{\left|K(k_1,k_2,k_3,k_4,k_5)\right|}{k_1 k_2 k_3 k_4 k_5} \leq \lim_{k_1,k_2,k_3,k_4,k_5 \to \infty} \frac{k_1^{\frac{1}{3}} k_2^{\frac{1}{3}} k_3^{\frac{1}{3}} k_4^{\frac{1}{3}} k_5^{\frac{1}{3}}}{k_1 k_2 k_3 k_4 k_5} = 0.$$

Definition 2.5: An *n*-sequence $\left(x_{k_1 \dots k_n}\right)$ is said to be statistically convergent to the number L if for each $\varepsilon > 0$,

$$\delta_n\left(\left\{\left(k_1,\dots,k_n\right) \in N \times \dots \times N : \left|x_{k_1 \dots k_n} - L\right| \geq \varepsilon\right\}\right) = 0.$$

If $\left(x_{k_1 \dots k_n}\right)$ is statistically convergent to the number L we denote this by

$$st - \lim_{k_1,\dots,k_n \to \infty} x_{k_1 \dots k_n} = L.$$

Remark 2.1: It is clear that if $\left(x_{k_1 \dots k_n}\right)$ is convergent then it is statistically convergent but the converse is not necessarily true.

Also a statistically convergent *n*-sequence need not be bounded which follows from the following example.

Example 2.3: Let us consider the 3-sequence $\left(x_{k_1 k_2 k_3}\right)$, where

$$x_{k_1 k_2 k_3} = \begin{cases} k_1 k_2 k_4, \text{when } k_1, k_2, k_3 \text{ are cubes} \\ 3, \text{otherwise.} \end{cases}$$

Then $st - \lim x_{k_1 k_2 k_3} = 3$, but $\left(x_{k_1 k_2 k_3}\right)$ is neither convergent in Pringsheim's sense nor bounded.

Definition 2.6: An *n*-sequence $\left(x_{k_1 k_2 \dots k_n}\right)$ is said to be statistically Cauchy sequence if for every $\varepsilon > 0$, there exist $l_i = l_i\left(\varepsilon\right) \in N,\ 1 \leq i \leq n$ such that

$$\delta_n\left(\left\{\left(k_1,\dots,k_n\right) \in N \times \dots \times N : \left|x_{k_1 \dots k_n} - x_{l_1 \dots l_n}\right| \geq \varepsilon\right\}\right) = 0.$$

Definition 2.7: Let $X = \left(x_{k_1 k_2 \dots k_n}\right)$ and $Y = \left(y_{k_1 k_2 \dots k_n}\right)$ be two *n*-sequences. Then we say that $x_{k_1 k_2 \dots k_n} = y_{k_1 k_2 \dots k_n}$ for almost all (a. a.) $k_1, k_2, \dots, k_n$ if

$$\delta_n\left(\left\{\left(k_1,\dots,k_n\right) \in N \times \dots \times N : x_{k_1 \dots k_n} \neq y_{k_1 \dots k_n}\right\}\right) = 0.$$

Definition 2.8: Let $X = \left(x_{k_1 k_2 \dots k_n}\right)$ be an *n*-sequence. A subset D of C, the set of complex numbers is said to contain $x_{k_1 k_2 \dots k_n}$ for almost all $k_1, k_2, \dots, k_n$ if

$$\delta_n\left(\left\{\left(k_1,\dots,k_n\right) \in N \times \dots \times N : x_{k_1 \dots k_n} \notin D\right\}\right) = 0.$$

3. STATISTICAL CONVERGENCE

Lemma 3.1: If $st - \lim_{k_1,\dots,k_n} x_{k_1 \dots k_n} = a$ and $st - \lim_{k_1,\dots,k_n} y_{k_1 \dots k_n} = b$ and c is a scalar, then

(i) $st - \lim_{k_1,\dots,k_n} \left(x_{k_1 \dots k_n} + y_{k_1 \dots k_n}\right) = a + b$.

(Ii) $st - \lim_{k_1,\dots,k_n} \left(c.x_{k_1 \dots k_n}\right) = ca$.

Proof: The proof is easy.

Proposition 3.1: If $\left(x_{k_1 k_2 \dots k_n}\right)$ is an n-sequence then $st - \lim_{k_1,\dots,k_n \to \infty} x_{k_1 \dots k_n} = L$ if and only if there exists a subset $K \subseteq N \times \dots \times N$ such that $\delta_n\left(K\right) = 1$ and

$$\lim_{\substack{k_1,\dots,k_n \to \infty \\ (k_1,\dots,k_n) \in K}} x_{k_1 \dots k_n} = L.$$

Proof: The proof follows from the proof of [6, Theorem 2].

Corollary 3.2: If $st - \lim_{k_1,\dots,k_n \to \infty} x_{k_1 \dots k_n} = L$ then there exists an n-sequence $y_{k_1 \dots k_n}$ such that $\lim_{k_1,\dots,k_n \to \infty} y_{k_1 \dots k_n} = L$ and

$$\delta_n\left(\left\{(k_1,...,k_n)\in N\times...\times N : x_{k_1...k_n}\neq y_{k_1...k_n}\right\}\right)=0 .$$

Theorem 3.3: An n-sequence $\left(x_{k_1k_2...k_n}\right)$ is statistically convergent if and only if it is a statistically Cauchy sequence.

Proof: Suppose $st-\lim_{k_1,...,k_n} x_{k_1...k_n}=l$ and $\varepsilon>0$. Then $\left|x_{k_1...k_n}-l\right|<\frac{\varepsilon}{2}$, for almost all $k_1,k_2,...,k_n\in N\times...\times N$ and we can choose $(m_1,m_2,...,m_n)\in N\times...\times N$ such that $\left|x_{m_1...m_n}-l\right|<\frac{\varepsilon}{2}$. Then we have

$$\left|x_{k_1...k_n}-x_{m_1...m_n}\right|\le\left|x_{k_1...k_n}-l\right|+\left|x_{m_1...m_n}-l\right|$$

$$<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon \text{, for almost all } k_1,k_2,...,k_n .$$

Hence $\left(x_{k_1k_2...k_n}\right)$ is a statistically Cauchy sequence

Next, assume $\left(x_{k_1k_2...k_n}\right)$ is a statistically Cauchy sequence and choose $\left(m_1^1,m_2^1,...,m_n^1\right)\in N\times...\times N$ so that the closed interval $J=\left[x_{m_1^1...m_n^1}-1,\ x_{m_1^1...m_n^1}+1\right]$ of length 2 contains $x_{k_1k_2...k_n}$ for almost all $k_1,k_2,...,k_n$. Again we can choose $\left(m_1^2,m_2^2,...,m_n^2\right)\in N\times...\times N$ so that the closed interval $J^{/}=\left[x_{m_1^2...m_n^2}-\frac{1}{2},\ x_{m_1^2...m_n^2}+\frac{1}{2}\right]$ of length 1 contains $x_{k_1k_2...k_n}$ for almost all $k_1,k_2,...,k_n$. If we take $J_1=J\cap J^{/}$, then J_1 is a closed interval of length less than or equal to 1 that contains $x_{k_1k_2...k_n}$ for almost all $k_1,k_2,...,k_n$. Now we proceed by choosing $\left(m_1^3,m_2^3,...,m_n^3\right)\in N\times...\times N$ so that $J^{//}=\left[x_{m_1^3...m_n^3}-\frac{1}{4},\ x_{m_1^3...m_n^3}+\frac{1}{4}\right]$ of length 1/2 contains $x_{k_1k_2...k_n}$ for almost all $k_1,k_2,...,k_n$. If we take $J_2=J_1\cap J^{//}$, then J_2 is a closed interval of length less than or equal to 1/2 that contains $x_{k_1k_2...k_n}$ for almost all $k_1,k_2,...,k_n$. Proceeding in this way inductively, we have a sequence (J_m) of closed intervals such that

(*i*) $J_{m+1}\subseteq J_m$, for all $m\in N$

(*ii*) Length $J_m\le 2^{1-m}$, for all $m\in N$

(*iii*) $x_{k_1k_2...k_n}\in J_m$ for almost all $k_1,k_2,...,k_n$ and for each $m\in N$.

Then by the nested interval theorem $\bigcap_{m=1}^{\infty}J_m$ contains one point. Denote this point by *v* and we shall show that $\left(x_{k_1k_2...k_n}\right)$ statistically convergent to *v*. Now $v\in J_m$, for all $m\in N$. If we choose *l* such that $\frac{1}{2^l}<\varepsilon$, then J_l contains $x_{k_1k_2...k_n}$ for almost all $k_1,k_2,...,k_n$. Hence we have $\left(x_{k_1k_2...k_n}\right)$ is a statistically convergent to *v*.

Theorem 3.4: Let X = $\left(x_{k_1k_2...k_n}\right)$ be an *n*-sequence. Then the following statements are equivalent:
(i) X is a statistically convergent sequence;
(ii) X is a statistically Cauchy sequence;
(iii) There exists a subsequence Y = $\left(y_{k_1k_2...k_n}\right)$ of X = $\left(x_{k_1k_2...k_n}\right)$ such that $x_{k_1k_2...k_n}$ = $y_{k_1k_2...k_n}$ for almost all $k_1,k_2,...,k_n$.

Proof: In view of the above theorem, the proof is easy.

Corollary 3.5: If X = $\left(x_{k_1k_2...k_n}\right)$ is an n-sequence such that $st-\lim_{k_1,...,k_n\to\infty}x_{k_1...k_n}=L$, then X has a subsequence Y = $\left(y_{k_1k_2...k_n}\right)$ such that $\lim_{k_1,...,k_n\to\infty}y_{k_1...k_n}=L$.

Theorem 3.6: The set $st\cap_n\ell_\infty$ of all bounded statistically convergent n-sequences is a closed linear subspace of the normed linear space ${}_n\ell_\infty$.

Proof: By Lemma 3.1, it is obvious that $st\cap_n\ell_\infty$ is a linear subspace of the normed linear space ${}_n\ell_\infty$. To prove the result it is sufficient to prove that $st\cap_n\ell_\infty$ is closed. Let $x^{(m_1m_2...m_n)}$ = $\left(x_{k_1k_2...k_n}^{(m_1m_2...m_n)}\right)$ be a convergent sequence in $st\cap_n\ell_\infty$ and converge to *x*. It is clear that $x\in{}_n\ell_\infty$. Since $x^{(m_1m_2...m_n)}\in st$, by definition of statistical convergence there exist real numbers $a_{m_1m_2...m_n}$ such that

$$st-\lim x_{k_1k_2...k_n}^{(m_1m_2...m_n)}=a_{m_1m_2...m_n}\ ,\ m_1,m_2,...,m_n=1,2,3,....$$

As $x^{(m_1m_2...m_n)}\to x$, this implies that $x^{(m_1m_2...m_n)}$ is a Cauchy sequence. So for each $\varepsilon>0$, there exists a positive integer n_0 such that

$$\left|x^{(p_1p_2...p_n)}-x^{(m_1m_2...m_n)}\right|<\frac{\varepsilon}{3},$$

for every $p_i\ge m_i\ge n_0$, $i=1,2,\ldots,n$

and |.| denotes the norm in the linear space. Since

$$st-\lim x_{k_1k_2...k_n}^{(m_1m_2...m_n)}=a_{m_1m_2...m_n}$$

and

$$st-\lim x_{k_1k_2...k_n}^{(p_1p_2...p_n)}=a_{p_1p_2...p_n}\ ,$$

by Proposition 3.1, there exists $K_1 \subseteq N \times ... \times N$ such that $\delta_n(K_1) = 1$ and

$$\lim_{\substack{k_1,...,k_n \to \infty \\ (k_1,...,k_n) \in K_1}} x_{k_1...k_n}^{(m_1...m_n)} = a_{m_1...m_n}$$

and there exists $K_2 \subseteq N \times ... \times N$ such that $\delta_n(K_2) = 1$

and $$\lim_{\substack{k_1,...,k_n \to \infty \\ (k_1,...,k_n) \in K_2}} x_{k_1...k_n}^{(p_1...p_n)} = a_{p_1...p_n}.$$

Since $\delta_n(K_1 \cap K_2) = 1$, $K_1 \cap K_2$ is not finite. Let us choose $(d_1,...,d_n) \in K_1 \cap K_2$ so that

$$\left| x_{d_1...d_n}^{(p_1...p_n)} - a_{p_1...p_n} \right| < \frac{\varepsilon}{3}$$

and

$$\left| x_{d_1...d_n}^{(m_1...m_n)} - a_{m_1...m_n} \right| < \frac{\varepsilon}{3}.$$

Hence for each $p_i \geq m_i \geq n_0$ ($i = 1, 2,\ldots,n$), we have

$$\left| a_{p_1...p_n} - a_{m_1...m_n} \right| \leq \left| x_{d_1...d_n}^{(m_1...m_n)} - a_{m_1...m_n} \right| + \left| x_{d_1...d_n}^{(p_1...p_n)} - x_{d_1...d_n}^{(m_1...m_n)} \right| + \left| x_{d_1...d_n}^{(p_1...p_n)} - a_{p_1...p_n} \right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

This implies that $\left(a_{m_1...m_n}\right)$ is a Cauchy sequence and consequently convergent. Let $\lim_{m_1...m_n} a_{m_1...m_n} = a$. Next our aim is to show that *x* is statistically convergent to *a*. Since $x^{(m_1 m_2 ... m_n)}$ is convergent to *x* in ${}_n\ell_\infty$, by the structure of ${}_n\ell_\infty$ it is also coordinate wise convergent. Therefore for each $\varepsilon > 0$, there exists a positive integer $n_1(\varepsilon)$ such that

$$\left| x_{k_1...k_n}^{(m_1...m_n)} - x_{k_1...k_n} \right| < \frac{\varepsilon}{3}, \text{ for every } m_1,...,m_n \geq n_1(\varepsilon)$$

and because $\lim_{m_1...m_n} a_{m_1...m_n} = a$, for each $\varepsilon > 0$, there exists $n_2(\varepsilon)$ such that

$$\left| a_{m_1...m_n} - a \right| < \frac{\varepsilon}{3}, \text{ for every } m_1,...,m_n \geq n_2(\varepsilon).$$

Finally since $x^{(m_1 m_2 ... m_n)}$ is statistically convergent to $a_{m_1 m_2 ... m_n}$, there exists $K \subseteq N \times ... \times N$ such that $\delta_n(K) = 1$

and $$\lim_{\substack{k_1,...,k_n \to \infty \\ (k_1,...,k_n) \in K}} x_{k_1...k_n}^{(m_1...m_n)} = a_{m_1...m_n}.$$

This means that for every $\varepsilon > 0$, there exists a positive integer $n_3(\varepsilon)$ such that

m $$\left| x_{k_1...k_n}^{(m_1...m_n)} - a_{m_1...m_n} \right| < \frac{\varepsilon}{3},$$

for every $m_1,...,m_n \geq n_3(\varepsilon)$ and $(k_1,...,k_n) \in K$.

Let $n_4(\varepsilon) = \max\{n_1(\varepsilon), n_2(\varepsilon), n_3(\varepsilon)\}$. Then

$$\left| x_{k_1...k_n} - a \right| \leq \left| x_{k_1...k_n}^{(m_1...m_n)} - x_{k_1...k_n} \right| + \left| x_{k_1...k_n}^{(m_1...m_n)} - a_{m_1...m_n} \right| + \left| a_{m_1...m_n} - a \right| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

So, *x* is statistically convergent to *a* and this completes the proof.

Corollary 3.7: The set $st \cap_n \ell_\infty$ is nowhere dense in ${}_n\ell_\infty$.

Proof: It is a well known fact that every closed linear subspace of an arbitrary linear normed space *E*, different from *E*, is a nowhere dense set in *E*. Hence on account of the above theorem it suffices to prove that $st \cap_n \ell_\infty \neq {}_n\ell_\infty$, which follows from the following example.

Example 3.1: Let *n* = 3 and consider the triple sequence $\left(x_{ijk}\right)$ defined as

$$x_{ijk} = \begin{cases} -3, & i, j, k \text{ are odd} \\ 3, & \text{otherwise.} \end{cases}$$

Then $\left(x_{ijk}\right)$ is bounded but not statistically convergent.

4. GENERALIZED KÖTHE-TOEPLITZ DUAL

The notion of α-duals is generalized by Chandra and Tripathy [3] by introducing the notion of η-duals of sequence spaces. Throughout the paper ${}_nw$, ${}_nc$, ${}_nc_0$, ${}_n\ell_1$, ${}_n\ell_p$, ${}_n\ell_\infty$, ${}_nbv$, ${}_n\sigma$ and ${}_nw_p$ denote the spaces of all, convergent in Pringsheim's sense, null in Pringsheim's sense, absolutely summable, p-absolutely summable, bounded, bounded variation, eventually alternating and strongly p-Cesàro summable n-sequence spaces respectively.

We have the following sequence spaces:

$${}_n\ell_\infty = \left\{ \left(a_{k_1...k_n}\right) \in {}_nw : \sup_{k_1,...,k_n} \left| a_{k_1...k_n} \right| < \infty \right\},$$

$${}_nc = \{(a_{k_1...k_n}) \in {}_nw : a_{k_1...k_n} \to L, \text{ as } \min(k_1,...,k_n) \to \infty, \text{ for some } L \in C\},$$

$${}_nc_0 = \left\{ \left(a_{k_1...k_n}\right) \in {}_nw : a_{k_1...k_n} \to 0, \text{as} \min\left(k_1,...,k_n\right) \to \infty \right\},$$

$${}_nbv = \{(a_{k_1...k_n}) \in {}_nw : \sum \left| \Delta_{k_1} a_{k_1...k_n} \right| < \infty, \ldots, \sum \left| \Delta_{k_n} a_{k_1...k_n} \right| < \infty \text{ and } \sum ... \sum \left| \Delta_{k_1...k_n} a_{k_1...k_n} \right| < \infty\},$$

where

$\Delta_{k_1} a_{k_1 \ldots k_n} = a_{k_1 \ldots k_n} - a_{k_1+1, k_2 \ldots k_n}, \ldots,$

$\Delta_{k_n} a_{k_1 \ldots k_n} = a_{k_1 \ldots k_n} - a_{k_1 \ldots k_{n-1}, k_n+1},$

$\Delta_{k_1, \ldots, k_n} a_{k_1 \ldots k_n} = \Delta_{k_2, \ldots, k_n} a_{k_1 k_2 \ldots k_n} - \Delta_{k_2, \ldots, k_n} a_{k_1+1, k_2 \ldots k_n}$ etc.

We define ${}_n bv_0 = {}_n bv \cap {}_n c_0$,

$${}_n w_p = \left\{ \left(a_{k_1 \ldots k_n}\right) \in {}_n w : \lim_{l_1, \ldots, l_n \to \infty} \frac{1}{l_1 \ldots l_n} \sum_{k_1=1}^{l_1} \ldots \sum_{k_n=1}^{l_n} \left| a_{k_1 \ldots k_n} - L \right|^p = 0 \right\}$$

$${}_n \sigma = \left\{ (a_{k_1 \ldots k_n}) \in {}_n w : a_{k_1 \ldots k_n} = -a_{k_1 \ldots k_{n-1}, k_n+1} \text{ for all } k_n \ge l_n, \ldots, a_{k_1 \ldots k_n} = -a_{k_1+1, k_2 \ldots k_n} \text{ for all } k_1 \ge l_1 \right\}.$$

Let E be a non-empty subset of ${}_n w$ and $r \ge 1$. Then the η-dual of E is defined as

$$E^{\eta} = \left\{ \left(a_{k_1 \ldots k_n}\right) \in {}_n w : \sum_{k_1} \ldots \sum_{k_n} \left| a_{k_1 \ldots k_n} b_{k_1 \ldots k_n} \right|^r < \infty \text{ for all} \left(b_{k_1 \ldots k_n}\right) \in E \right\}$$

The space E is said to be η-reflexive if $E^{\eta\eta} = E$. Taking $r = 1$ in the above definition we get the α-dual (Köthe-Toeplitz dual) of E, *i.e.*, E^{α}, for $E \subset {}_n w$.

The proof of the following results is obvious in view of the definition of η-dual of *n*-sequences.

Lemma 4.1: Let E and F be any two non-empty subsets of ${}_n w$. Then

(*i*) E^{η} is a linear subspace of ${}_n w$.

(*ii*) $E \subset F$ implies $F^{\eta} \subset E^{\eta}$.

(*iii*) $E \subseteq E^{\eta\eta}$.

Theorem 4.1: $\left({}_n \ell_r\right)^{\eta} = {}_n \ell_{\infty}$ and $\left({}_n \ell_{\infty}\right)^{\eta} = {}_n \ell_r$. The spaces ${}_n \ell_r$ and ${}_n \ell_{\infty}$ are perfect spaces.

Proof. Let $\left(a_{k_1 \ldots k_n}\right) \in {}_n \ell_{\infty}$. Then we have

$$\sum_{k_1} \ldots \sum_{k_n} \left| a_{k_1 \ldots k_n} b_{k_1 \ldots k_n} \right|^r < \infty \quad \text{for all} \quad \left(b_{k_1 \ldots k_n}\right) \in {}_n \ell_r.$$

Hence

$${}_n \ell_{\infty} \subseteq \left({}_n \ell_r\right)^{\eta}.$$

Conversely let $\left(a_{k_1 \ldots k_n}\right) \notin {}_n \ell_{\infty}$. Then there exists sequence of positive integers $\left(l_{i_1}\right), \ldots, \left(l_{i_n}\right)$ such that

$$a_{l_{i_1} \ldots l_{i_n}} > i.$$

Define the *n*-sequence $\left(b_{k_1 \ldots k_n}\right)$ as follows

$$b_{k_1 \ldots k_n} = i^{-1}, \text{if } k_1 = l_{i_1}, \ldots, k_n = l_{i_n}$$

$$= 0, \text{ otherwise.}$$

Then $\left(b_{k_1 \ldots k_n}\right) \in {}_n \ell_r$, but $\left(a_{k_1 \ldots k_n} b_{k_1 \ldots k_n}\right) \notin {}_n \ell_r$.

Hence $\left({}_n \ell_r\right)^{\eta} \subseteq {}_n \ell_{\infty}$.

The proof for the case $\left({}_n \ell_{\infty}\right)^{\eta} = {}_n \ell_r$ is a routine work. This completes the proof of the Theorem.

Theorem 4.2: $\left({}_n bv\right)^{\eta} = \left({}_n bv_0\right)^{\eta} = {}_n \ell_r$. The spaces ${}_n bv$ and ${}_n bv_0$ are not perfect.

Proof: We have ${}_n bv_0 \subseteq {}_n \ell_{\infty}$. Hence we have

$${}_n \ell_r = \left({}_n \ell_{\infty}\right)^{\eta} \subseteq \left({}_n bv_0\right)^{\eta}.$$

Next we show that

$$\left({}_n bv_0\right)^{\eta} \subseteq {}_n \ell_r.$$

Let $\left(b_{k_1 \ldots k_n}\right) \notin {}_n \ell_r$. Then we can find a sequence $\left(l_i\right)$ of positive integers with $l_1 = 1$ such that

$$\sum_{k_1=1}^{\infty} \ldots \sum_{k_{n-1}=1}^{\infty} \sum_{k_n=l_i}^{l_{i+1}-1} \left| b_{k_1 \ldots k_n} \right|^r > i^r \text{ for all } i = 1, 2, \ldots$$

Define $\left(a_{k_1 \ldots k_n}\right)$ as follows:

$$a_{k_1 \ldots k_n} = i^{-1}, \text{if } l_i \le k_n < l_{i+1}, \text{ for all } i = 1, 2, \ldots$$

Then

$$\sum_{k_1=1}^{\infty} \ldots \sum_{k_n=1}^{\infty} \left| \Delta a_{k_1 \ldots k_n} \right| = \sum_{k_1=1}^{\infty} \ldots \sum_{i=1}^{\infty} \left(\sum_{k_n=l_i}^{l_{i+1}-1} \left| \Delta a_{k_1 \ldots k_n} \right| \right)$$

$$= \sum_{k=1}^{\infty} \ldots \sum_{i=1}^{\infty} \left(\sum_{k_n=l_i}^{l_{i+1}-1} \left| a_{k_1 \ldots k_n} - a_{k_1 \ldots k_{n-1}, k_n+1} - a_{k_1 \ldots k_{n-1}+1, k_n} + \ldots + a_{k_1 \ldots k_{n-1}+1, k_n+1} - \ldots - a_{k_1+1 \ldots k_n+1} \right| \right)$$

$$= \sum_{k_1=1}^{\infty} \ldots \sum_{i=1}^{\infty} \left(\sum_{k_n=l_i}^{l_{i+1}-1} \left| \frac{1}{i} - \frac{1}{i+1} + \ldots + \frac{1}{i} - \frac{1}{i+1} \right| \right)$$

$$= 0.$$

Hence $\left(a_{k_1 \ldots k_n}\right) \in {}_n bv_0$.

Now

$$\sum_{k_1=1}^{\infty} \ldots \sum_{k_n=1}^{\infty} \left| a_{k_1 \ldots k_n} b_{k_1 \ldots k_n} \right|^r = \sum_{i=1}^{\infty} \sum_{k_1=1}^{\infty} \ldots \sum_{k_{n-1}=1}^{\infty} \sum_{k_n=l_i}^{l_{i+1}-1} \left| a_{k_1 \ldots k_n} b_{k_1 \ldots k_n} \right|^r$$

$$= \sum_{i=1}^{\infty} \frac{1}{i^r} \sum_{k_1=1}^{\infty} \ldots \sum_{k_{n-1}=1}^{\infty} \sum_{k_n=l_i}^{l_{i+1}-1} \left| b_{k_1 \ldots k_n} \right|^r$$

$$> \sum_{i=1}^{\infty} \frac{1}{i^r} i^r$$

$= \infty$, a contradiction.

Hence $\left({}_n bv_0\right)^{\eta} \subseteq {}_n \ell_r$.

Thus we have

$$\left({}_n bv_0\right)^\eta = {}_n\ell_r .$$

The proof of $\left({}_n bv\right)^\eta = {}_n\ell_r$ follows from the following inclusion

$${}_n bv_0 \subseteq {}_n bv \subseteq {}_n\ell_\infty .$$

Hence it follows from the theorem 4.1 that the spaces ${}_n bv$ and ${}_n bv_0$ are not perfect.

Theorem 4.3: $\left({}_n\sigma\right)^\eta = {}_n\ell_r$. The space ${}_n\sigma$ is not perfect.

Proof: We have ${}_n\sigma \subseteq {}_n\ell_\infty$.
Hence

$${}_n\ell_r = \left({}_n\ell_\infty\right)^\eta \subseteq \left({}_n\sigma\right)^\eta .$$

For converse part, let $\left(b_{k_1\ldots k_n}\right) \in \left({}_n\sigma\right)^\eta$. Then

$$\sum_{k_1}\cdots\sum_{k_n}\left|a_{k_1\ldots k_n} b_{k_1\ldots k_n}\right|^r < \infty \text{ for all } \left(a_{k_1\ldots k_n}\right) \in {}_n\sigma .$$

Consider $a_{k_1\ldots k_n} = 1 = -a_{k_1+1\ldots k_n} = \ldots = -a_{k_1\ldots k_n+1}$, for all $k_1,\ldots,k_n \in N$. Then

$$\left(a_{k_1\ldots k_n}\right) \in {}_n\sigma$$

and

$$\sum_{k_1}\cdots\sum_{k_n}\left|b_{k_1\ldots k_n}\right|^r < \infty .$$

This implies that

$$\left(b_{k_1\ldots k_n}\right) \in {}_n\ell_r .$$

Hence

$$\left({}_n\sigma\right)^\eta \subseteq {}_n\ell_r .$$

Thus $\left({}_n\sigma\right)^\eta = \ell_r^n$.

Hence it follows from Theorem 4.1 that the space ${}_n\sigma$ is not perfect.

Theorem 4.4: $\left({}_n w_p \cap {}_n\ell_\infty\right)^\eta = {}_n \ell_r$. The space ${}_n w_p \cap {}_n\ell_\infty$ is not perfect.

Proof: Clearly ${}_n\ell_r \subseteq \left({}_n w_p \cap {}_n\ell_\infty\right)^\eta$.

Conversely, let $\left(a_{k_1\ldots k_n}\right) \notin {}_n\ell_r$. Then we can write

$$\sum_{k_1}\cdots\sum_{k_n}\left|a_{k_1\ldots k_n}\right|^r = \infty .$$

Consider the *n*-sequence $\left(b_{k_1\ldots k_n}\right)$, defined by

$$b_{k_1\ldots k_n} = j, \text{ a constant, for all } k_1,\ldots,k_n \in N .$$

Then $\left(b_{k_1\ldots k_n}\right) \in {}_n w_p \cap {}_n\ell_\infty$, but

$$\sum_{k_1}\cdots\sum_{k_n}\left|a_{k_1\ldots k_n} b_{k_1\ldots k_n}\right|^r = \infty .$$

Hence

$$\left(a_{k_1\ldots k_n}\right) \notin \left({}_n w_p \cap {}_n\ell_\infty\right)^\eta .$$

It follows that

$$\left({}_n w_p \cap {}_n\ell_\infty\right)^\eta \subseteq {}_n\ell_r .$$

Thus $\left({}_n w_p \cap {}_n\ell_\infty\right)^\eta = {}_n\ell_r$.

Hence it follows from Theorem 4.1 that the space ${}_n w_p \cap {}_n\ell_\infty$ is not perfect.

ACKNOWLEDGEMENT

The authors are grateful to the referee for careful reading of the article and suggested improvements.

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