Statistical convergence of \( n \)-sequences and \( \eta \)-dual of some classical sets of \( n \)-sequences

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ABSTRACT

In this paper we introduce the notion of \( n \)-sequence and extend the notion of statistical convergence to \( n \)-sequences. Further we define the notion of \( \eta \)-dual as a generalization of Köthe-Toeplitz dual for subsets of \( n \)-sequence spaces and compute \( \eta \)-dual of some classical sets of \( n \)-sequences.

\[ \text{|n-sequence| statistical convergence| completeness| Köthe-Toeplitz dual|} \]

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1. INTRODUCTION

Pringsheim gave the definition of the convergence for double sequences in 1900. Since then, this concept has been studied by many authors, see for instances [7, 14, 21]. The notion of the statistical convergence was first independently introduced by Fast [4] in 1951 and Schoenberg [17] in 1959. Later on it was further investigated from a sequence space point of view and linked with summability theory by Fridy [5], Salat [18] and many others. In [12] and [13] the above concept is extended to double sequences by using the idea of a two dimensional analogue of natural density.

It is a fundamental principle of functional analysis that investigations of spaces are often combined with those of dual spaces. The notion of duals of sequence spaces was introduced by Köthe and Toeplitz [9]. Later on it was studied by Maddox [11], Lascarides [10], Bektaş, Et and Çolak [2], Chandra and Tripathy [3], Sarma [19], Dutta [1] and many others.

2. DEFINITIONS AND PRELIMINARIES

Now we introduce some basic notions and examples related to the subject of this paper.

**Definition 2.1:** Let \( n(\geq 2) \) be an integer. A function \( x : N \times \ldots \times N(n - \text{factors}) \rightarrow R(C) \) is called a real (complex) \( n \)-sequence, where \( N, R \) and \( C \) denote the sets of natural numbers, real numbers and complex numbers respectively.

**Example 2.1:** Consider the 4-sequence \( \left( x_{k_1,k_2,k_3,k_4} \right) \), where

\[
\begin{cases}
  k_2k_3k_4k_1 = 2 \\
  k_1k_3k_4k_2 = 4 \\
  k_1k_2k_4k_3 = 6 \\
  k_1k_2k_3k_4 = 8 \\
  10, \text{otherwise.}
\end{cases}
\]

Then \( \left( x_{k_1,k_2,k_3,k_4} \right) \) converges to 10 in Pringsheim’s sense.
Definition 2.3: An $n$-sequence $\left(x_{k_1...k_n}\right)$ is said to be a Cauchy sequence if for every $\varepsilon > 0$, there exists $M(\varepsilon) \in N$ such that $\left|x_{k_1...k_n} - x_{m_1...m_n}\right| < \varepsilon$, whenever $k_i \geq m_i \geq M, i = 1,...,n$.

Definition 2.4: An $n$-sequence $\left(x_{k_1...k_n}\right)$ is said to be bounded if there exists $U > 0$ such that $\left|x_{k_1...k_n}\right| < U$ for all $k_i, i = 1,...,n$.

We denote the set of all bounded $n$-sequences by $\ell^\infty$. It is easy to show that $\ell^\infty$ is a normed space, normed by

$$\|x\|_{\ell^\infty} = \sup_{k_1,..,k_n} \left|x_{k_1...k_n}\right|.$$ 

A subset $K$ of $N \times ... \times N (n-factors)$ is said to have natural density $\delta_n(K)$ if

$$\delta_n(K) = \lim_{k_1,...,k_n \to \infty} \frac{|K(k_1,...,k_n)|}{k_1...k_n}.$$ 

Here $|K(k_1,...,k_n)|$ denotes the numbers of $(l_1,...,l_n)$ in $K$ such that $l_i \leq k_i, i = 1,...,n$.

Example 2.2: Consider the set

$$K = \left\{ (l_1, l_2, l_3, l_4, l_5) : l_1, l_2, l_3, l_4, l_5 \in N \right\} \subseteq N \times N \times N \times N \times N.$$ 

Then

$$\delta_5(K) = \lim_{k_1, k_2, k_3, k_4, k_5 \to \infty} \frac{|K(k_1, k_2, k_3, k_4, k_5)|}{k_1 k_2 k_3 k_4 k_5} = \lim_{k_1, k_2, k_3, k_4, k_5 \to \infty} \frac{k_1 k_2 k_3 k_4 k_5}{k_1 k_2 k_3 k_4 k_5} = 0.$$ 

Definition 2.5: An $n$-sequence $\left(x_{k_1...k_n}\right)$ is said to be statistically convergent to the number $L$ if for each $\varepsilon > 0$,

$$\delta_n\left(\left\{ (k_1,...,k_n) \in N \times ... \times N : \left|x_{k_1...k_n} - L\right| \geq \varepsilon \right\}\right) = 0.$$ 

If $\left(x_{k_1...k_n}\right)$ is statistically convergent to the number $L$ we denote this by

$$st-\lim_{k_1,...,k_n \to \infty} x_{k_1...k_n} = L.$$ 

Remark 2.1: It is clear that if $\left(x_{k_1...k_n}\right)$ is convergent then it is statistically convergent but the converse is not necessarily true.

Also a statistically convergent $n$-sequence need not be bounded which follows from the following example.

Example 2.3: Let us consider the 3-sequence $\left(x_{k_1k_2k_3}\right)$, where

$$x_{k_1k_2k_3} = \begin{cases} k_1k_2k_3, & \text{when } k_1, k_2, k_3 \text{ are cubes} \\ 3, & \text{otherwise} \end{cases}.$$ 

Then $st-\lim x_{k_1k_2k_3} = 3$, but $\left(x_{k_1k_2k_3}\right)$ is neither convergent in Pringsheim’s sense nor bounded.

Definition 2.6: An $n$-sequence $\left(x_{k_1k_2...k_n}\right)$ is said to be statistically Cauchy sequence if for every $\varepsilon > 0$, there exist $l_i = l_i(\varepsilon) \in N, 1 \leq i \leq n$ such that

$$\delta_n\left(\left\{ (k_1,...,k_n) \in N \times ... \times N : \left|x_{k_1...k_n} - x_{l_1...l_n}\right| \geq \varepsilon \right\}\right) = 0.$$ 

Definition 2.7: Let $X = \left(x_{k_1k_2...k_n}\right)$ and $Y = \left(y_{k_1k_2...k_n}\right)$ be two $n$-sequences. Then we say that $x_{k_1k_2...k_n} = y_{k_1k_2...k_n}$ for almost all (a. a.) $k_1, k_2,...,k_n$ if

$$\delta_{2n}\left(\left\{ (k_1,...,k_n) \in N \times ... \times N : x_{k_1...k_n} \neq y_{k_1...k_n}\right\}\right) = 0.$$ 

Definition 2.8: Let $X = \left(x_{k_1k_2...k_n}\right)$ be an $n$-sequence. A subset $D$ of $C$, the set of complex numbers is said to contain $x_{k_1k_2...k_n}$ for almost all $k_1, k_2,...,k_n$ if

$$\delta_{2n}\left(\left\{ (k_1,...,k_n) \in N \times ... \times N : x_{k_1...k_n} \notin D\right\}\right) = 0.$$ 

3. STATISTICAL CONVERGENCE

Lemma 3.1: If $st-\lim_{k_1,...,k_n \to \infty} x_{k_1...k_n} = a$ and

$$st-\lim_{k_1,...,k_n \to \infty} y_{k_1...k_n} = b$$

and $c$ is a scalar, then

(i) $st-\lim_{k_1,...,k_n \to \infty} \left(x_{k_1...k_n} + y_{k_1...k_n}\right) = a + b$.

(ii) $st-\lim_{k_1,...,k_n \to \infty} \left(c x_{k_1...k_n}\right) = ca$.

Proof: The proof is easy.

Proposition 3.1: If $\left(x_{k_1k_2...k_n}\right)$ is an $n$-sequence then

$$st-\lim_{k_1,...,k_n \to \infty} x_{k_1...k_n} = L$$

if and only if there exists a subset $K \subseteq N \times ... \times N$ such that $\delta_n(K) = 1$ and

$$\lim_{k_1,...,k_n \to \infty} x_{k_1...k_n} = L.$$ 

Proof: The proof follows from the proof of [6, Theorem 2].

Corollary 3.2: If $st-\lim_{k_1,...,k_n \to \infty} x_{k_1...k_n} = L$ then there exists an $n$-sequence $y_{k_1...k_n}$ such that $\lim_{k_1,...,k_n \to \infty} y_{k_1...k_n} = L$ and
\[ \delta_n \left( \{ (k_1, \ldots, k_n) \in N \times \ldots \times N : x_{k_1 \ldots k_n} \neq y_{k_1 \ldots k_n} \} \right) = 0. \]

**Theorem 3.3:** An n-sequence \( (x_{k_1 \ldots k_n}) \) is statistically convergent if and only if it is a statistically Cauchy sequence.

**Proof:** Suppose \( st - \lim_{k_1 \ldots k_n} x_{k_1 \ldots k_n} = l \) and \( \varepsilon > 0 \). Then
\[ \left| x_{k_1 \ldots k_n} - l \right| < \frac{\varepsilon}{2}, \text{ for almost all } k_1, k_2, \ldots, k_n \in N \times \ldots \times N \]
and we can choose \( (m_1, m_2, \ldots, m_n) \in N \times \ldots \times N \) such that
\[ \left| x_{m_1 \ldots m_n} - l \right| < \frac{\varepsilon}{2}. \]
Then we have
\[ |x_{k_1 \ldots k_n} - x_{m_1 \ldots m_n}| \leq |x_{k_1 \ldots k_n} - l| + |x_{m_1 \ldots m_n} - l| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \text{ for almost all } k_1, k_2, \ldots, k_n. \]
Hence \( (x_{k_1 \ldots k_n}) \) is a statistically Cauchy sequence.

Next, assume \( (x_{k_1 \ldots k_n}) \) is a statistically Cauchy sequence and choose \( (m_1, m_2^1, \ldots, m_n^1) \in N \times \ldots \times N \) so that the closed interval \( J = \left[ x_{m_1^1 \ldots m_n^1} - 1, x_{m_1^1 \ldots m_n^1} + 1 \right] \) of length 2 contains \( x_{k_1 \ldots k_n} \) for almost all \( k_1, k_2, \ldots, k_n \). Again we can choose \( (m_1^2, m_2^2, \ldots, m_n^2) \in N \times \ldots \times N \) so that the closed interval \( J' = \left[ x_{m_1^2 \ldots m_n^2} - \frac{1}{2}, x_{m_1^2 \ldots m_n^2} + \frac{1}{2} \right] \) of length 1 contains \( x_{k_1 \ldots k_n} \) for almost all \( k_1, k_2, \ldots, k_n \). If we take \( J_1 = J \cap J' \), then \( J_1 \) is a closed interval of length less than or equal to 1 that contains \( x_{k_1 \ldots k_n} \) for almost all \( k_1, k_2, \ldots, k_n \). Now we proceed by choosing \( (m_1^3, m_2^3, \ldots, m_n^3) \in N \times \ldots \times N \) so that
\[ J'' = \left[ x_{m_1^3 \ldots m_n^3} - \frac{1}{4}, x_{m_1^3 \ldots m_n^3} + \frac{1}{4} \right] \]
of length 1/2 contains \( x_{k_1 \ldots k_n} \) for almost all \( k_1, k_2, \ldots, k_n \). If we take \( J_2 = J_1 \cap J'' \), then \( J_2 \) is a closed interval of length less than or equal to 1/2 that contains \( x_{k_1 \ldots k_n} \) for almost all \( k_1, k_2, \ldots, k_n \). Proceeding in this way inductively, we have a sequence \( (J_m) \) of closed intervals such that
(i) \( J_{m+1} \subseteq J_m \), for all \( m \in N \)
(ii) Length \( J_m \leq 2^{-m} \), for all \( m \in N \)
(iii) \( x_{k_1 \ldots k_n} \in J_m \) for almost all \( k_1, k_2, \ldots, k_n \) and for each \( m \in N \).

Then by the nested interval theorem \( \bigcap_{m=1}^{\infty} J_m \) contains one point. Denote this point by \( v \) and we shall show that \( (x_{k_1 \ldots k_n}) \) statistically convergent to \( v \). Now \( v \in J_m \), for all \( m \in N \). If we choose \( l \) such that \( \frac{1}{2^l} < \varepsilon \), then \( J_l \) contains \( x_{k_1 \ldots k_n} \) for almost all \( k_1, k_2, \ldots, k_n \). Hence we have \( (x_{k_1 \ldots k_n}) \) is a statistically convergent to \( v \).

**Theorem 3.4:** Let \( X = (x_{k_1 \ldots k_n}) \) be an n-sequence. Then the following statements are equivalent:
(i) \( X \) is a statistically convergent sequence;
(ii) \( X \) is a statistically Cauchy sequence;
(iii) There exists a subsequence \( Y = (y_{k_1 \ldots k_n}) \) of \( X \) such that \( y_{k_1 \ldots k_n} = x_{k_1 \ldots k_n} \) for almost all \( k_1, k_2, \ldots, k_n \).

**Proof:** In view of the above theorem, the proof is easy.

**Corollary 3.5:** If \( X = (x_{k_1 \ldots k_n}) \) is an n-sequence such that \( st - \lim_{k_1 \ldots k_n} x_{k_1 \ldots k_n} = L \), then \( X \) has a subsequence \( Y = (y_{k_1 \ldots k_n}) \) such that \( \lim_{k_1 \ldots k_n} y_{k_1 \ldots k_n} = L \).

**Theorem 3.6:** The set \( st \cap_{n} \ell_{\infty} \) of all bounded statistically convergent n-sequences is a closed linear subspace of the normed linear space \( n \ell_{\infty} \).

**Proof:** By Lemma 3.1, it is obvious that \( st \cap_{n} \ell_{\infty} \) is a linear subspace of the normed linear space \( n \ell_{\infty} \). To prove the result it is sufficient to prove that \( st \cap_{n} \ell_{\infty} \) is closed. Let \( x(m_{m_1 \ldots m_n}) = (x(m_{m_1 \ldots m_n})) \) be a convergent sequence in \( st \cap_{n} \ell_{\infty} \) and converge to \( x \). It is clear that \( x \in n \ell_{\infty} \). Since \( x(m_{m_1 \ldots m_n}) \in st \), by definition of statistical convergence there exist real numbers \( a_{m_1 \ldots m_n} \) such that
\[ st - \lim x(m_{m_1 \ldots m_n}) = a_{m_1 \ldots m_n}, m_1, m_2, \ldots, m_n = 1, 2, 3, \ldots \]
As \( x(m_{m_1 \ldots m}) \rightarrow x \), this implies that \( x(m_{m_1 \ldots m}) \) is a Cauchy sequence. So for each \( \varepsilon > 0 \), there exists a positive integer \( n_0 \) such that
\[ \left| x(p_i \ldots p_n) - x(m_{m_1 \ldots m_n}) \right| < \frac{\varepsilon}{3}, \]
for every \( p_i \geq m_i, \geq n_0, i = 1, 2, \ldots, n \)
and \( |\cdot| \) denotes the norm in the linear space. Since
\[ st - \lim x(m_{m_1 \ldots m_n}) = a_{m_1 \ldots m_n} \]
and
\[ st - \lim x(p_i \ldots p_n) = a_{p_i \ldots p_n}, \]
next, for every sequence \( x_{k_1 \ldots k_n} \).
by Proposition 3.1, there exists $K_1 \subseteq N \times \ldots \times N$ such that $\delta_n(K_1) = 1$ and

$$\lim_{(k_i, \ldots, k_n) \to \infty} x_{k_1, \ldots, k_n} = a_{m_1 \ldots m_n},$$

and there exists $K_2 \subseteq N \times \ldots \times N$ such that $\delta_n(K_2) = 1$ and

$$\lim_{(k_i, \ldots, k_n) \to \infty} x_{(p_i, p_n)} = a_{p_i \ldots p_n}.$$

Since $\delta_n(K_1 \cap K_2) = 1$, $K_1 \cap K_2$ is not finite. Let us choose $(d_1, \ldots, d_n) \in K_1 \cap K_2$ so that

$$|x_{d_1, \ldots, d_n} - a_{p_i \ldots p_n}| < \frac{\varepsilon}{3},$$

and

$$|x_{(m_i \ldots m_n)} - a_{m_1 \ldots m_n}| < \frac{\varepsilon}{3}.$$

Hence for each $p_i \geq m_i \geq n_0$ ($i = 1, 2, \ldots, n$), we have

$$|a_{p_i \ldots p_n} - a_{m_1 \ldots m_n}| \leq |x_{d_1, \ldots, d_n} - a_{m_1 \ldots m_n}| +$$

$$+ |x_{(p_i \ldots p_n)} - x_{(m_i \ldots m_n)}| + |x_{d_1, \ldots, d_n} - x_{(p_i \ldots p_n)}| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

This implies that $(a_{m_1 \ldots m_n})$ is a Cauchy sequence and consequently convergent. Let $\lim_{m_1 \ldots m_n} a_{m_1 \ldots m_n} = a$. Next our aim is to show that $x$ is statistically convergent to $a$. Since $x_{(m_1 \ldots m_n)}$ is convergent to $x$ in $\ell_\infty$, by the structure of $\ell_\infty$, it is also coordinate wise convergent. Therefore for each $\varepsilon > 0$, there exists a positive integer $n_1(\varepsilon)$ such that

$$|x_{k_1 \ldots k_n} - a_{m_1 \ldots m_n}| < \frac{\varepsilon}{3},$$

for every $m_1 \ldots m_n \geq n_1(\varepsilon)$ and because $\lim_{m_1 \ldots m_n} a_{m_1 \ldots m_n} = a$, for each $\varepsilon > 0$, there exists $n_2(\varepsilon)$ such that

$$|a_{m_1 \ldots m_n} - a| < \frac{\varepsilon}{3},$$

for every $m_1 \ldots m_n \geq n_2(\varepsilon)$.

Finally since $x_{(m_1 \ldots m_n)}$ is statistically convergent to $a_{m_1 \ldots m_n}$, there exists $K \subseteq N \times \ldots \times N$ such that $\delta_n(K) = 1$ and

$$\lim_{(k_i, \ldots, k_n) \to \infty} x_{k_1 \ldots k_n} = a_{m_1 \ldots m_n}.$$

This means that for every $\varepsilon > 0$, there exists a positive integer $n_3(\varepsilon)$ such that

$$m |x_{k_1 \ldots k_n} - a_{m_1 \ldots m_n}| < \frac{\varepsilon}{3},$$

for every $m_1 \ldots m_n \geq n_3(\varepsilon)$ and $(k_1, \ldots, k_n) \in K$.

Let $n(\varepsilon) = \max\{n_1(\varepsilon), n_2(\varepsilon), n_3(\varepsilon)\}$. Then

$$|x_{k_1 \ldots k_n} - a| \leq |x_{k_1 \ldots k_n} - x_{k_1 \ldots k_n} + x_{k_1 \ldots k_n} - a_{m_1 \ldots m_n}| +$$

$$+ |a_{m_1 \ldots m_n} - a| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

So, $x$ is statistically convergent to $a$ and this completes the proof.

**Corollary 3.7:** The set $st \cap_n \ell_\infty$ is nowhere dense in $\ell_\infty$.

**Proof:** It is a well known fact that every closed linear subspace of an arbitrary linear normed space $E$, different from $E$, is a nowhere dense set in $E$. Hence on account of the above theorem it suffices to prove that $st \cap_n \ell_\infty \not= \ell_\infty$, which follows from the following example.

**Example 3.1:** Let $n = 3$ and consider the triple sequence $(x_{ijk})$ defined as

$$x_{ijk} = \begin{cases} -3, & i, j, k \text{ are odd} \\ 3, & \text{otherwise.} \end{cases}$$

Then $(x_{ijk})$ is bounded but not statistically convergent.

4. **GENERALIZED KÖTHE-TOEPLITZ DUAL**

The notion of $\alpha$-duals is generalized by Chandra and Tripathy [3] by introducing the notion of $\eta$-duals of sequence spaces. Throughout the paper $w_n$, $c_n$, $e_0$, $\ell_1$, $\ell_p$, $\ell_\infty$, $bv_n$, $\sigma_n$ and $w_p$ denote the spaces of all, convergent in Pringsheim’s sense, null in Pringsheim’s sense, absolutely summable, p-absolutely summable, bounded, bounded variation, eventually alternating and strongly p-Cesàro summable n-sequence spaces respectively.

We have the following sequence spaces:

$$\tilde{\ell}_\infty = \left\{ (a_{k_1 \ldots k_n})_n : \sup_{k_1, \ldots, k_n} a_{k_1 \ldots k_n} < \infty \right\},$$

$$\eta = \left\{ (a_{k_1 \ldots k_n})_n : a_{k_1 \ldots k_n} \to 0 \text{, as } \min(k_1, \ldots, k_n) \to \infty \text{, for some } L \in C \right\},$$

$$c_0 = \left\{ (a_{k_1 \ldots k_n})_n : a_{k_1 \ldots k_n} \to 0, \text{ as } \min(k_1, \ldots, k_n) \to \infty \right\},$$

$$bv_n = \left\{ (a_{k_1 \ldots k_n})_n : \sum_{k_1} a_{k_1 \ldots k_n} < \infty, \ldots, \sum_{k_1} \sum_{k_2} \ldots \sum_{k_n} a_{k_1 \ldots k_n} < \infty \right\},$$

where

$$x_{k_1 \ldots k_n} - a = \left( a_{m_1 \ldots m_n} - a_{m_1 \ldots m_n} \right) \leq \left( a_{m_1 \ldots m_n} - a_{m_1 \ldots m_n} \right) +$$

$$+ \left( a_{m_1 \ldots m_n} - a \right) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$
\[ \Delta_{k_1} a_{k_1 \ldots k_n} = a_{k_1 \ldots k_n} - a_{k_1+1, k_2, \ldots, k_n} \ldots, \]
\[ \Delta_{k_1} a_{k_1 \ldots k_n} = a_{k_1 \ldots k_n} - a_{k_1, k_2, \ldots, k_{n+1}}, \]
\[ \Delta_{k_1 \ldots k_n} a_{k_1 \ldots k_n} = \Delta_{k_2, \ldots, k_n} a_{k_1 k_2 \ldots k_n} - \Delta_{k_2, \ldots, k_n} a_{k_1+1, k_2, \ldots, k_n} \text{ etc.} \]

We define \( n^w_p = n^w \cap c_0 \),
\[ n^w_p = \left\{ \left( a_{k_1 \ldots k_n} \right) \in n^w : \lim_{k_i \to \infty} \frac{1}{L} \sum_{k_i=1}^{L} \sum_{k_{i-1}=1}^{L} \left| a_{k_1 \ldots k_n} - L \right|^{p} = 0 \right\} \]
\[ n^\sigma = \left\{ \left( a_{k_1 \ldots k_n} \right) \in n^w : a_{k_1 \ldots k_n} = -a_{k_1 \ldots k_{n-1} k_{n+1}} \text{ for all } k_n > l_n, \ldots, a_{k_1 \ldots k_n} = -a_{k_1 k_{n-1} k_{n+1}} \text{ for all } k_1 \geq l_1 \right\}. \]

Let \( E \) be a non-empty subset of \( n^w \) and \( r \geq 1 \).

Then the \( \eta \)-dual of \( E \) is defined as
\[ E^\eta = \left\{ \left( a_{k_1 \ldots k_n} \right) \in n^w : \sum_{k_1}^{\infty} \sum_{k_n}^{\infty} \left| a_{k_1 \ldots k_n} b_{k_1 \ldots k_n} \right|^{p} < \infty \text{ for all } \left( b_{k_1 \ldots k_n} \right) \in E \right\} \]

The space \( E \) is said to be \( \eta \)-reflexive if \( E^\eta = E \).

Taking \( r = 1 \) in the above definition we get the \( \alpha \)-dual (Köthe-Toeplitz dual) of \( E \), i.e., \( E^\alpha \), for \( E \subset n^w \).

The proof of the following results is obvious in view of the definition of \( \eta \)-dual of \( n \)-sequences.

**Lemma 4.1:** Let \( E \) and \( F \) be any two non-empty subsets of \( n^w \). Then
(i) \( E^\eta \) is a linear subspace of \( n^w \).
(ii) \( E \subset F \) implies \( F^\eta \subset E^\eta \).
(iii) \( E \subset E^\eta \).

**Theorem 4.1:** \( (n^\ell_r)^\eta = n^\ell_\infty \) and \( (n^\ell_\infty)^\eta = n^\ell_r \). The spaces \( n^\ell_r \) and \( n^\ell_\infty \) are perfect spaces.

**Proof.** Let \( \left( a_{k_1 \ldots k_n} \right) \in n^\ell_\infty \). Then we have
\[ \sum_{k_1}^{\infty} \sum_{k_n}^{\infty} \left| a_{k_1 \ldots k_n} b_{k_1 \ldots k_n} \right|^{p} < \infty \text{ for all } \left( b_{k_1 \ldots k_n} \right) \in n^\ell_r. \]

Hence
\[ n^\ell_\infty \subset (n^\ell_r)^\eta. \]

Conversely let \( \left( a_{k_1 \ldots k_n} \right) \in n^\ell_\infty \). Then there exists sequence of positive integers \( (l_1), \ldots, (l_i) \) such that
\[ a_{l_1 \ldots l_i} > i. \]

Define the \( n \)-sequence \( \left( b_{k_1 \ldots k_n} \right) \) as follows
\[ b_{k_1 \ldots k_n} = i^{-1}, \text{ if } k_1 = l_1, \ldots, k_n = l_n \]
\[ = 0, \text{ otherwise.} \]
Then \( \left( b_{k_1 \ldots k_n} \right) \in n^\ell_r, \text{ but } \left( a_{k_1 \ldots k_n} b_{k_1 \ldots k_n} \right) \notin n^\ell_r. \]

Hence \( (n^\ell_r)^\eta \subset n^\ell_\infty \).

The proof for the case \( (n^\ell_\infty)^\eta = n^\ell_r \) is a routine work. This completes the proof of the Theorem.

**Theorem 4.2:** \( (n^w)^\eta = (n^w_0)^\eta = n^\ell_r \). The spaces \( n^w \) and \( n^w_0 \) are not perfect.

**Proof:** We have \( n^w_0 \subset (n^\ell_\infty)^\eta \). Hence we have
\[ n^\ell_r = (n^\ell_\infty)^\eta \subset (n^w_0)^\eta. \]

Next we show that \( (n^w_0)^\eta \subset n^\ell_r \).

Let \( \left( b_{k_1 \ldots k_n} \right) \notin n^\ell_r \). Then we can find a sequence \( (l_i) \) of positive integers with \( l_i = 1 \) such that
\[ \sum_{k_1}^{\infty} \sum_{k_n}^{\infty} \left| b_{k_1 \ldots k_n} \right|^{p} > i^r \text{ for all } i = 1, 2, \ldots \]

Define \( \left( a_{k_1 \ldots k_n} \right) \) as follows:
\[ a_{k_1 \ldots k_n} = i^{-1}, \text{ if } k_1 \leq k_n < l_{i+1}, \text{ for all } i = 1, 2, \ldots \]

Then
\[ \sum_{k_1}^{\infty} \sum_{k_n}^{\infty} \left| a_{k_1 \ldots k_n} \right|^{p} = \sum_{k_1}^{\infty} \sum_{k_{i-1}}^{\infty} \left( \sum_{k_{i-1}}^{\infty} \left| a_{k_{i-1} \ldots k_n} \right|^{p} \right), \]
\[ = \sum_{k_1}^{\infty} \sum_{k_{i-1}}^{\infty} \left( \left| a_{k_{i-1} \ldots k_n} \right|^{p} - \left| a_{k_{i-1} \ldots k_{n+1} \ldots k_n} \right|^{p} \right), \]
\[ = \sum_{k_1}^{\infty} \sum_{k_{i-1}}^{\infty} \left( \frac{1}{i^r} + \ldots + \frac{1}{i+1} \right), \]
\[ = 0. \]

Hence \( (n^w_0)^\eta \subset n^\ell_r \).

Thus we have
\[ (n^w_0)^\eta \subset n^\ell_r \].
The proof of \((n \mathbf{b}_V)^\sigma = n \ell_r\) follows from the following inclusion
\[ n \mathbf{b}_V \subseteq n \mathbf{b} \subseteq n \ell_\infty. \]
Hence it follows from the theorem 4.1 that the spaces \(n \mathbf{b}\) and \(n \mathbf{b}_V\) are not perfect.

**Theorem 4.3:** \((n \sigma)^\sigma = n \ell_r\). The space \(n \sigma\) is not perfect.

**Proof:** We have \(n \sigma \subseteq n \ell_\infty\). Hence
\[ n \ell_r = (n \mathbf{b})^\sigma \subseteq (n \sigma)^\sigma. \]
For converse part, let \((b_{k_1 \ldots k_n}) \in (n \sigma)^\sigma\). Then
\[ \sum_{k_1}^{\ldots} \sum_{k_n}^{\ldots} |a_{k_1 \ldots k_n} b_{k_1 \ldots k_n}| < \infty \] for all \((a_{k_1 \ldots k_n}) \in n \sigma\).

Consider \(a_{k_1 \ldots k_n} = 1 = -a_{k_1 +1 \ldots k_n} = \ldots = -a_{k_1 \ldots k_n+1}\), for all \(k_1, \ldots, k_n \in N\). Then
\[ (a_{k_1 \ldots k_n}) \in n \sigma \]
and
\[ \sum_{k_1}^{\ldots} \sum_{k_n}^{\ldots} |b_{k_1 \ldots k_n}| < \infty. \]
This implies that
\[ (b_{k_1 \ldots k_n}) \in n \ell_r. \]
Hence
\[ (n \sigma)^\sigma \subseteq n \ell_r. \]
Thus \((n \sigma)^\sigma = n \ell_r\).

Hence it follows from Theorem 4.1 that the space \(n \sigma\) is not perfect.

**Theorem 4.4:** \((n w_p \cap n \ell_\infty)^\sigma = n \ell_r\). The space \(n w_p \cap n \ell_\infty\) is not perfect.

**Proof:** Clearly \(n \ell_r \subseteq (n w_p \cap n \ell_\infty)^\sigma\).

Conversely, let \((a_{k_1 \ldots k_n}) \not\in n \ell_r\). Then we can write
\[ \sum_{k_1}^{\ldots} \sum_{k_n}^{\ldots} |a_{k_1 \ldots k_n} b_{k_1 \ldots k_n}| = \infty. \]

Consider the \(n\)-sequence \((b_{k_1 \ldots k_n})\), defined by
\[ b_{k_1 \ldots k_n} = j, \text{ a constant, for all } k_1, \ldots, k_n \in N. \]
Then \((b_{k_1 \ldots k_n}) \in n w_p \cap n \ell_\infty\), but
\[ \sum_{k_1}^{\ldots} \sum_{k_n}^{\ldots} |a_{k_1 \ldots k_n} b_{k_1 \ldots k_n}| = \infty. \]
Hence
\[ (a_{k_1 \ldots k_n}) \not\in (n w_p \cap n \ell_\infty)^\sigma. \]
It follows that
\[ (n w_p \cap n \ell_\infty)^\sigma = n \ell_r. \]

Thus \((n w_p \cap n \ell_\infty)^\sigma = n \ell_r\).

Hence it follows from Theorem 4.1 that the space \(n w_p \cap n \ell_\infty\) is not perfect.

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**REFERENCES**


