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RESEARCH ARTICLE

Constrained for G^1 Cubic Trigonometric Spline Curve Interpolation for Positive Data Set

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Abstract This paper presented the G^1 cubic trigonometric spline function with three shape parameters that generate a constrained curve interpolates 2D data. This research ensures that the generated curve passes through all data points in a positive data set yet satisfies the three cases of line constraints given. The three cases are: the data must lie above line L_i, the data must lie below line L_i, and lastly, the data must lie between two lines L_i^1 and L_i^2. Simpler schemes with less computation are implemented involving the roles of shape parameters. Two of the shape parameters are set free, while another parameter is fixed to fulfil all the three cases stated. The results show that a smooth curve of the G^1 cubic trigonometric spline function can be produced within the constrained line by using the schemes developed while the hereditary shape of the data is preserved. Numerical examples are illustrated and discussed.

Keywords: constrained curve, cubic trigonometric spline, geometric continuity, interpolation.

Introduction

Preserving and constraining data shape is essential in dealing with interpolation for data visualization. Constrained data interpolation will help the designer design a curve that follows the given data points drawn within the barriers and limitations. The interpolation curve is used to generate actual data and is also involved in designing cars, houses, medical images, robot paths or other industrial designs. The presented interpolation must carry the hereditary features of regular data. Works of literature show many studies in interpolating and preserving data positivity, and constrained data interpolation.

It is essential to preserve the positivity of data since it has been used in many areas such as rainfall distribution, heart rate, and population [1]. Most researchers choose to study quadratic and cubic trigonometric spline. It is due to the simplicity in the formulation and can give a smooth representation of the data. The performance of the developed curves is tested using some constraints such as lines.

[2] developed constraints for Quadratic and Cubic Bezier curves by determining the middle control points to avoid the curve crossing the constrained line. [3] gave an approach to derive constraints to preserve the shape of the data using a trigonometric interpolant. The resulting curves show it preserves the data's shape but did not generate smooth curves. [4] produced theorem for interpolation of C^2 quintic Hermite polynomial curve that lies above the constrained line that preserved the shape of the data.

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Some researchers have extended the formulation to rational form to increase the flexibility in controlling the fitted curve. [5] have derived restriction to visualize curve and surface lying above the line and above the plane on free parameters. The result from the produced scheme showed it is local and computationally economical. They also extended the rational cubic curve to a rational bi-cubic partially blended surface to preserve the data's surface shape. [6] constrained C^1 continuous rational quadratic Bezier curve drawn within a closed boundary of straight line segments. [7] constructed C^1-rational cubic fractal interpolation function (RCFIF) in preserving the constrained aspect of the data. [8] implemented scheme for preserving positivity shape of the data that may lie above, below and between the constrained line. It was discovered that this scheme does not produce the constrained interpolating curve that lies between two straight lines. Their scheme has been improved by [9], in which they performed constrained interpolation using rational cubic spline with three parameters. The produced interpolating curve lies below or above an arbitrary straight line or between two straight lines.

Although there are mushrooming techniques in developing trigonometric spline curves, there is room for improvement. Factors such as the continuity, the number of shape parameters used, the curve's degrees, and the number of basis functions should be considered to generate a better result. Therefore, this paper proposes a scheme wherein a constrained interpolation curve lies below or above an arbitrary line or lies between two straight lines using a cubic trigonometric spline curve developed by [1] that satisfies G^1 continuity condition. The developed scheme requires minimal computation time since the spline is in a non-rational form. This paper considers a cubic trigonometric spline that consists of three shape parameters to give more freedom to the generated curve. These shape parameters occur when joining the piecewise curves with G^1 continuity. The formulation of this trigonometric spline curve is explained in Section 2. Section 3 discusses the behavior of this curve in interpolating positive data within certain constraints. The results are presented in Section 4. This paper ends with a conclusion in Section 5.

G^1 Cubic Trigonometric Spline Curve Interpolation

The definition of G^1 cubic trigonometric spline curve interpolation in this paper is originated by [1]. The given 2D set of data points (x_i, f_i), $i = 1,2,\ldots,n$, where $x_1 < x_2 < \ldots < x_n$. f_i are the function values and let d_i be at the endpoints of the first derivatives. Let $\theta = \frac{\pi}{2}u_i$, $u_i = \frac{x-x_i}{h_i}$, $h_i = x_{i+1} - x_i$. The G^1 cubic trigonometric spline function with two shape parameters β_1 and β_2, are defined over each subinterval $[x_i, x_{i+1}]$ as:

$$P_i(x) = \sum_{l=0}^{2} A_l(\theta) B_l(\theta), \forall x \epsilon\, [x_i, x_{i+1}], l = 0,1,2. \tag{1}$$

where

$$A_0(\theta) = (1 - \sin\theta)^2\,(1 - \beta_1 sin\theta)$$
$$A_1(\theta) = \left(1 - A_0(\theta) - A_2(\theta)\right)$$
$$A_2(\theta) = (1 - \cos\theta)^2\,(1 - \beta_2 cos\theta)$$

are the cubic trigonometric basis functions. By applying the G^1 conditions [10],

$$P(x_i) = f_i, \qquad P(x_{i+1}) = f_{i+1}, \qquad P'(x_i) = \gamma_i d_i \qquad P'(x_{i+1}) = d_{i+1},$$

After some calculations, the control points are :

$$B_0 = f_i$$
$$B_1 = \frac{2h_i}{\pi\beta_1 + 2\pi}\gamma_i d_i + f_i\,;$$
$$B_2 = f_{i+1}\,; \tag{2}$$

G^1 Shape Preserving Constrained Data

In a previous paper [1], we generated a cubic trigonometric spline curve which preserved the shape of positive data. In this section, a curve construction technique is presented which interpolates data not only considering lying at the position $y = 0$ axis but also should be above, below or between line or constrained $f_i = mx_i + c$. We implemented schemes that give the results of curve that interpolates data besides constrained by using two constrain lines. Let the positive data set $\{(x_i, f_i); i = 0,1,2, \dots, n\}$ lies above the straight line $y = mx + c$ where m is the gradient of the line and c is the intercept of the y-axis line. This hypothesis is expressed mathematically in equation (3).

$$f_i > mx_i + c \tag{3}$$

where cubic trigonometric spline curve will lie above the arbitrary straight line if $P_i(x) > mx_i + c$,$\forall x \in [a, b]$ and a, b are the points of the straight line, L_i.

For each subinterval $[x_i, x_{i+1}]$, by expressing the straight line in terms of parameter θ, the equation (3) takes the following parametric form,

$$L_i = a_i\left(1 - \frac{2}{\pi}\theta\right) + b_i\frac{2}{\pi}\theta,$$

$$a_i = mx_i + c, \qquad b_i = mx_{i+1} + c$$

Case 1: Constrained data above line

Data will lies above the straight line $y = mx + c$ if and only if

$$P_i(x) > L_i(\theta).$$

$$P_i(x) - L_i(\theta) > 0 \tag{4}$$

Substituting (1) in (4), we have

$$\sum_{l=0}^{2} A_l(x)B_l(\theta) - L_i(\theta) > 0 \tag{5}$$

since $\sum_{l=0}^{2} A_l(x) = 1$, (5) becomes

$$\sum_{l=0}^{2} A_l(x)B_l(\theta) - \sum_{l=0}^{2} A_l(x)L_i(\theta) = \sum_{l=0}^{2} A_l(x)Q_l(\theta) > 0 \tag{6}$$

where

$$Q_0 = f_i - L_i, \qquad Q_1 = \left((f_i - L_i) + \frac{2h_i d_i \gamma_i}{\pi\beta_1 + 2\pi}\right), \qquad Q_2 = f_{i+1} - L_i,$$

Since $A_l(x) > 0, l = 0,1,2,3$, relation (5) holds only if $Q_l(\theta) > 0, l = 0,1,2$. Thus, the sufficient conditions for the interpolating curve to lie above the straight line are as shown below:

$$Q_0 > 0 \text{ when } f_i > L_i$$

$$Q_1 > 0 \text{ when } \beta_1 > -2\left(\frac{h_i d_i \gamma_i}{\pi(f_i - L_i)} + 1\right) \quad \text{and}$$

$$Q_1 > 0 \text{ when } f_{i+1} > L_i$$

Thus, the curve drawn by using G^1 trigonometric spline will lie above a straight line if

$$\beta_1 > max\left\{0, -2\left(\frac{h_i d_i \gamma_i}{\pi(f_i - L_i)} + 1\right)\right\} \tag{7}$$

The derived condition on the parameters can also be written as:

$$\beta_1 = e_i + max\left\{0, -2\left(\frac{h_i d_i \gamma_i}{\pi(f_i - L_i)} + 1\right)\right\}, \quad e_i, \gamma_i > 0.$$

Case 2: Constrained data below line

Data will lies below the straight line $y = mx + c$ if and only if

$$L_i(\theta) > P_i(x)$$

$$L_i(\theta) - P_i(x) > 0 \tag{8}$$

Substituting (1) in (8), we have

$$L_i(\theta) - \sum_{l=0}^{2} A_l(x) B_l(\theta) > 0 \tag{9}$$

since $\sum_{l=0}^{2} A_l(x) = 1$, (9) becomes

$$\sum_{l=0}^{2} A_l(x) L_i(\theta) - \sum_{l=0}^{2} A_l(x) Q_l(\theta) > 0 \tag{10}$$

where

$$Q_0 = L_i - f_i, \qquad Q_1 = \left((L_i - f_i) + \frac{2h_i d_i \gamma_i}{\pi\beta_1 + 2\pi}\right), \qquad Q_2 = L_i - f_{i+1},$$

Since $A_l(x) > 0, l = 0,1,2,3$, relation (9) holds only if $Q_l(\theta) > 0, l = 0,1,2,3$. Thus, the sufficient conditions for the interpolating curve to lie above the straight line are as shown below:

$$Q_0 > 0 \text{ when } L_i > f_i$$

$$Q_1 > 0 \text{ when } \beta_1 > -2\left(\frac{h_i d_i \gamma_i}{\pi(L_i - f_i)} + 1\right) \quad \text{and}$$

$$Q_1 > 0 \text{ when } L_i > f_{i+1}$$

Thus, the curve drawn by using G^1 trigonometric spline will lie below a straight line if

$$\beta_1 > max\left\{0, -2\left(\frac{h_i d_i \gamma_i}{\pi(L_i - f_i)} + 1\right)\right\} \tag{11}$$

The derived condition on the parameters can also be written as:

$$\beta_1 = r_i + max\left\{0, -2\left(\frac{h_i d_i \gamma_i}{\pi(L_i - f_i)} + 1\right)\right\}, \quad r_i, \gamma_i > 0.$$

Case 3: Constrained data between two lines

Data will lies between the two straight line $y_1 = m_1 x + c_1$ and $y_2 = m_2 x + c_2$ if and only if

$$L_i^1(\theta) < P_i(x) < L_i^2(\theta), \tag{12}$$

where $y_1 = m_1 x + c_1$

$$L_i^1 = a_i^1\left(1-\frac{2}{\pi}\theta\right) - b_i^1\frac{2}{\pi}\theta, \quad a_i^1 = m_1x_i + c_1, \quad b_i^1 = m_1x_{i+1} + c_1$$

$$L_i^2 = a_i^2\left(1-\frac{2}{\pi}\theta\right) - b_i^2\frac{2}{\pi}\theta, \quad a_i^2 = m_2x_i + c_2, \quad b_i^2 = m_2x_{i+1} + c_2$$

The relation (12) implies that

$$L_i^1(\theta) < P_i(x), \ \forall_i = 0,1,2,\dots,n-1 \tag{13}$$

$$P_i(x) < L_i^2(\theta) \quad \forall_i = 0,1,2,\dots,n-1 \tag{14}$$

Solving (13) gives,

$$0 < P_i(x) - L_i^1, \ \forall_i = 0,1,2,\dots,n-1 \tag{15}$$

$$\sum_{l=0}^{2} A_l(x)B_l(\theta) - L_i^1 > 0$$

since $\sum_{i=0}^{2} A_l(x) = 1$, (15) becomes

$$\textstyle\sum_{l=0}^{2} A_i(x)B_l - \sum_{i=0}^{2} A_l(x)L_i^1(\theta) = \sum_{i=0}^{2} A_l(x)Q_i^*(\theta) > 0 \tag{16}$$

where

$$Q_0^* = f_i - L_i^1 \qquad Q_1^* = \left((f_i - L_i^*) + \tfrac{2h_id_i\gamma_i}{\pi\beta_1+2\pi}\right) \qquad Q_2^* = f_{i+1} - L_{i+1}^1$$

Since $A_l(x) > 0, l = 0,1,2$, relation (10) holds only if $Q_i^*(\theta) > 0, l = 0,1,2$. Obviously,

$$Q_0^* > 0 \quad \text{and} \quad Q_2^* > 0$$

$$Q_1^* > 0 \ \text{ if } \ \beta_1 > -2\left(\tfrac{h_id_i\gamma_i}{\pi(f_i-L_i^1)} + 1\right) \tag{17}$$

Solving (14) gives,

$$L_i^2(\theta) - P_i(x) > 0, \qquad \forall_i = 0,1,2,n-1 \tag{18}$$

$$L_i^2 - \sum_{l=0}^{2} A_l(x)B_l(\theta) > 0$$

Since $\sum_{l=0}^{2} A_l(x) = 1$, relation (18) becomes

$$\textstyle\sum_{l=0}^{2} A_l(x)L_i^2(\theta) - \sum_{l=0}^{2} A_l(x)B_l(\theta) = \sum_{l=0}^{2} A_l(x)R_i^*(\theta) > 0 \tag{19}$$

where

$$R_0^* = L_i^2 - f_i\,, \qquad R_1^* = \left((L_i^2 - f_i) + \tfrac{2h_id_i\gamma_i}{\pi\beta_1+2\pi}\right), \quad R_2^* = L_{i+1}^2 - f_{i+1}$$

since $A_l(x) > 0, l = 0,1,2$, relation (19) holds only if $R_i^* > 0, i = 0,1,2$. Obviously,

$$R_0^* > 0 \text{ and } R_2^* > 0$$

and

$$R_1^* > 0 \text{ if } \beta_1 > -2\left(\frac{h_i d_i \gamma_i}{\pi(L_i^2 - f_i)} + 1\right) \quad ; \gamma_i > 0 \tag{20}$$

Combining the conditions (17) and (20) gives

$$\beta_1 \ > \max\{0, a_1, a_2\} \ ; \ \gamma_i > 0$$

where

$$a_1 = -2\left(\frac{h_i d_i \gamma_i}{\pi(f_i - L_i^1)} + 1\right), \quad a_2 = -2\left(\frac{h_i d_i \gamma_i}{\pi(L_i^2 - f_i)} + 1\right)$$

The derived condition on the parameters can also be written as,

$$\beta_1 = v_i + max\left\{0, -2\left(\frac{h_i d_i \gamma_i}{\pi(f_i - L_i^1)} + 1\right), -2\left(\frac{h_i d_i \gamma_i}{\pi(L_i^2 - f_i)} + 1\right)\right\}, \quad v_i, \gamma_i > 0.$$

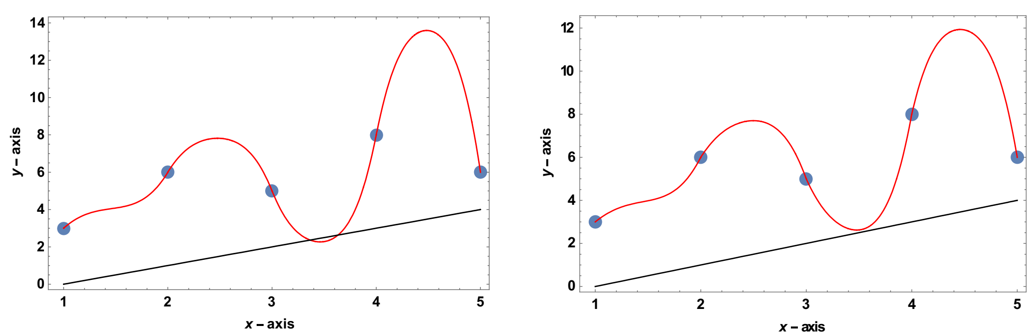
Results and discussion

In this section, we used the proposed schemes with the positive data set taken from [10] as shown in Table 1.

Table 1. Positive data from [11]

i	1	2	3	4	5
x_i	1	2	3	4	5
f_i	3	6	5	8	6

Case 1: Constrained Data Above Line



(a) (b)

Figure 1. (a) Non-constrained cubic trigonometric spline (b) Constrained cubic trigonometric spline

Table 2. Values of the shape parameter β_1, β_2 and γ_1

Shape parameter	Before constrained				After constrained			
i	1	2	3	4	1	2	3	4
$\beta_{1(i)}$	1.0	1.0	1.0	1.0	0.6	0.6	1.64892	0.6
$\beta_{2(i)}$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
γ_i	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0

Fig. 1 demonstrates the curves drawn for the function values listed in Table 1. All the data points lie above the line $L = \frac{4}{5}x,$ but Fig. 1(a) clearly shows the cubic trigonometric spline does not lie above the given straight line. This means that the required shape of the data presented in Table 1 is not achieved. In Fig. 1(b), the curve is generated by applying the constrained curve interpolation scheme (7). The values of free shape parameters β_2 and γ_i are selected randomly and the value of β_1 is constrained. All the values of shape parameters stated in Table 2. The resulting curve in Fig. 1 (b) shows that it is continuous, smooth and lies above the line L.

Case 2: Constrained Data Below Line

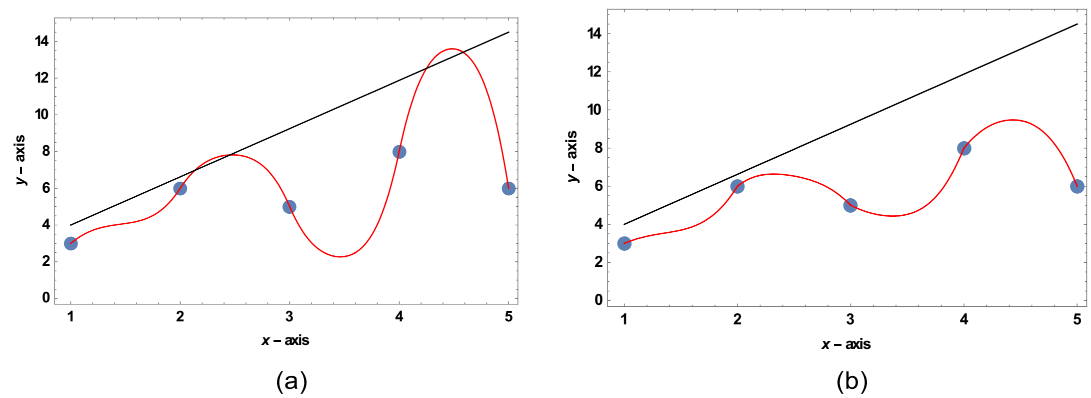


Figure 2. (a) Non-constrained cubic trigonometric spline (b) Constrained cubic trigonometric spline

Table 3. Values of shape parameter β_1, β_2 and γ_1

Shape parameter	Before constrained				After constrained			
i	1	2	3	4	1	2	3	4
$\beta_{1(i)}$	1.0	1.0	1.0	1.0	1.14648	10.3234	7.01225	0.6
$\beta_{2(i)}$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
γ_i	1.0	1.0	1.0	1.0	0.8	0.8	0.8	0.8

Fig. 2(a) presents a curve drawn by using G^1 cubic trigonometric spline for random values of parameters. The curve in Fig. 2(a) does not lie below the given straight line $L = 2.625x + 0.875$. Though the curve seems to be smooth, it does not carry the shape feature of the data. However, this flaw has been recovered nicely by using the proposed scheme (11). In Fig. 2(b), the constrained interpolating curve is generated by constraining the values of β_1, and the chosen value of free parameters β_2 and γ_i are shown in Table 3. The cubic trigonometric spline generated preserves the required shape of the given data. The data lies below the given straight line.

Case 3: Constrained Data Between Two Lines

In Fig. 3, G^1 cubic trigonometric spline is developed together with two given straight lines. The curve in Fig. 3 (a) shows that it does not lie above $L = \frac{4}{5}x$ and below $L = 2.625x + 0.875$, line and it does not retain the data shape. This drawback is amended by implementing the constrained curve scheme in (21), with the value of the free parameters $\beta_2 = 1.0$ and $\gamma_i = 0.8$ for the entire piecewise curves. The value of β_1 is constrained and fixed. The resulting values of β_1 after being constrained using (21) are shown in Table 4. Using the scheme developed in (21), we obtained the interpolating curve that lies above and below straight lines and inherits the given data's shape, as illustrated in Fig. 3(b).

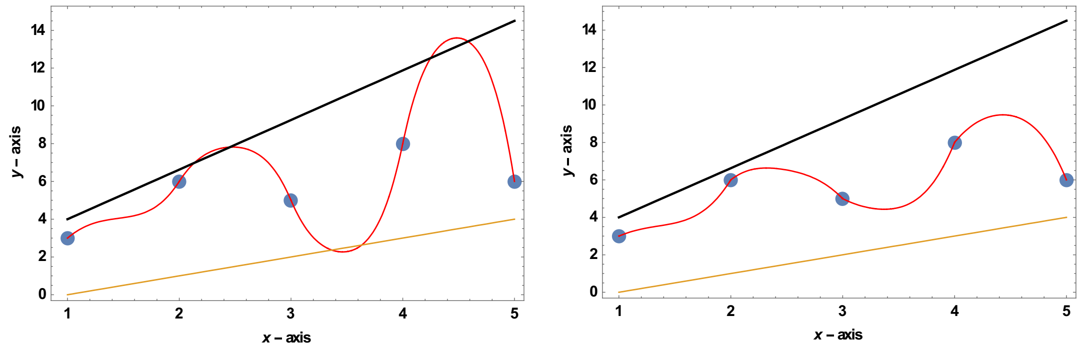


Figure 3. (a) Non-constrained cubic trigonometric spline (b) Constrained cubic trigonometric spline

Table 4. Values of shape parameter β_1, β_2 and γ_1

Shape parameter	Before constrained				After constrained			
i	1	2	3	4	1	2	3	4
$\beta_{1(i)}$	1.0	1.0	1.0	1.0	1.14648	7.01225	0.6	1.36331
$\beta_{2(i)}$	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
γ_i	1.0	1.0	1.0	1.0	0.8	0.8	0.8	0.8

Conclusions

The interpolation scheme of G^1 cubic trigonometric spline specifically, a curve constrained by line is proposed in this paper for 2D data to carry the built-in shape and features of the data. Three shape parameters are included in the spline for this paper. The value of one of the shape parameters is restricted while the other two parameters are free. The proposed interpolant is based on cubic trigonometric functions that appear more flexible and generate smoother graphical results than polynomial functions. The generation of smooth curves is vital in getting effective and good results. This work can be extended to a higher continuity or by increasing the number of restricted shape parameters to form a reliable curve.

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