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RESEARCH ARTICLE

Lazy Cops and Robber on Certain Cayley Graph

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Abstract The lazy cop number is the minimum number of cops needed for the cops to have a winning strategy in the game of Cops and Robber where at most one cop may move in any one round. This variant of the game of Cops and Robber, called Lazy Cops and Robber, was introduced by Offner and Ojakian, who provided bounds for the lazy cop number of hypercubes. In this paper, we are interested in the game of Lazy Cops and Robber on a Cayley graph of n copies of $\mathbf{Z}_{m+1}$.

Keywords: Cops and Robber, vertex-pursuit games, hypercubes.

Introduction

The game of Cops and Robber is a well-known two-player game played on a finite connected undirected graph. It was independently introduced by Quilliot [10], and Nowakowski and Winkler [8]. Initially, the first player occupies some vertices with some number of cops (multiple cops may occupy a single vertex) and then the second player occupies a vertex with a single robber. After that the cops and robber move alternatively along the edges of the graph. On the cops' turn, each of the cops may remain stationary or move to an adjacent vertex. On the robber's turn, he may remain stationary or move to an adjacent vertex. The move of a cop followed by the move of the robber constitutes one *round* of the game. The cops win if after a finite number of rounds, one of them can move to *catch* the robber, that is, the cop and the robber occupy the same vertex. In the game of Cops and Robber, the main objective is to study the *cop number*, i.e. the minimum number of cops required to catch the robber, introduced by Aigner and Fromme [1]. The most famous unsolved question in this context is Meyniel's conjecture [7]: the cop number of a connected graph with n vertices is $O(\sqrt{n})$.

Many variants of Cops and Robber have been studied. See [5, 12, 14, 13] for some of the related problems. We are interested in a variant introduced by Offner and Ojakian [9], where at most one cop moves in any one round. It is called the game of *Lazy Cops and Robber* and the *lazy cop number* is the minimum number of cops required to catch the robber in this setting. Let $c_L(G)$ be the lazy cop number of a graph G. Offner and Ojakian [9] gave lower and upper bounds for the lazy cop number of the hypercube. The lower bound was later improved by Bal, Bonato, Kinnersley, and Pralat [3], by using the probabilistic method coupled with a potential function argument. They also studied the game of Lazy Cops and Robber on random graphs and graphs on surfaces [4].

Cop number of a Cayley graph has been studied by Frankl [7]. We are interested in the lazy cop number of the Cayley graph of

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$$\Gamma(m,n) = \overbrace{\mathbb{Z}_{m+1} \times \mathbb{Z}_{m+1} \times \cdots \times \mathbb{Z}_{m+1}}^{n}$$

with generating set

$$S = \{\pm \mathbf{e}_i : 1 \le i \le n\},$$

where

$$\mathbf{e}_i = (0,\ldots,0,\overbrace{1}^{i\text{-th term}},0,\ldots,0).$$

Here, $\mathbb{Z}_{m+1} = \{0,1,2,\ldots,m\}$. Note that $\Gamma(m,n)$ is an abelian group. In fact, if $\mathbf{u},\mathbf{v} \in \Gamma(m,n)$ with

$$\mathbf{u} = (u_1,u_2,\ldots,u_n),$$
$$\mathbf{v} = (v_1,v_2,\ldots,v_n),$$

then, $\mathbf{u}+\mathbf{v} = (u_1+v_1,u_2+v_2,\ldots,u_n+v_n) \in \Gamma(m,n)$, where the operation in each coordinate is taken modulo $m+1$. The Cayley graph of $\Gamma(m,n)$ is the graph with vertex set $\Gamma(m,n)$ and $\mathbf{u},\mathbf{v} \in \Gamma(m,n)$ are adjacent to each other if and only if

$$\mathbf{u} = \mathbf{v} + \mathbf{s} \quad \text{for some } \mathbf{s} \in S.$$

This means that if $\mathbf{u}$ and $\mathbf{v}$ differ at coordinate i-th and they are adjacent, then

$$\mathbf{v} = (u_1,u_2,\ldots,u_{i-1},u_i \pm 1,u_{i+1},\ldots,u_n).$$

So, $\deg(\mathbf{u}) = 2n$ for all $\mathbf{u} \in \Gamma(m,n)$ and the Cayley graph is a $2n$-regular graph. We shall denote the Cayley graph by $\Gamma(m,n)$.

Note that when $m=1$, the graph $\Gamma(1,n)$ is a hypercube Q_n. When $m=2$, the graph $\Gamma(2,n)$ is a special case of the generalized hypercube considered in [11].

The main result of this study is to give a lower bound and an upper bound for the lazy cop number of $\Gamma(m,n)$. They are established in the next two sections.

Upper bound

Given a graph with vertices u,v, the *distance* between u and v is the number of edges in a shortest path connecting them. A *dominating set* of an undirected graph with vertex set V is a set $U \subseteq V$ such that every vertex in V Ç U has at least one neighbour in U. It is clear that by occupying a dominating set of a graph, the cops win. So the lazy cop number (and also the cop number) of a graph is bounded above by the size of the smallest dominating set of the graph.

Lemma 2.1 [2, Theorem 1.2.2 on p. 6] *A graph with N vertices and minimum degree $\delta > 1$ has a dominating set of size at most $N\dfrac{1+\ln(1+\delta)}{1+\delta}$.*

Theorem 2.2 *Let m,n be positive integers. If $n \ge 7$, then,*

$$c_L(\Gamma(m,n)) \le \frac{3(m+1)^{n-1}\ln n}{n}.$$

Proof. Let G_k denote the subgraph of $\Gamma(m,n)$ with vertex set

$$\{(k,x_2,\ldots,x_n) : x_i \in \mathbb{Z}_{m+1}\}.$$

That is, the subgraph generated by all the vertices with the first coordinate equal to k. Note that it has $(m+1)^{n-1}$ vertices and each vertex is of degree $2(n-1)$. By Lemma 2.1, G_k has a dominating set of size at most

$$(m+1)^{n-1}\frac{1+\ln(1+2(n-1))}{1+2(n-1)} = (m+1)^{n-1}\frac{1+\ln(2n-1)}{2n-1} \leq (m+1)^{n-1}\frac{\ln n}{n},n \quad (1)$$

where the last inequality follows from $n \geq 7$.

We have $\frac{3(m+1)^{n-1}\ln n}{n}$ cops that can be used. Initially, the cops can dominate G_0, G_1 and G_2. Cops occupying G_0 will be coloured with black, cops occupying G_1 will be coloured with blue, and cops occupying G_2 will be coloured with red. If the robber chooses an initial position with the first coordinate equal to $0,1$ or 2, then he will be caught immediately. So, the robber has to choose a position with first coordinate not equal to $0,1$ or 2. At first, the black and red cops should remain in place, while the blue cops rearrange themselves one by one to dominate G_3. During this process, the robber cannot have moved to the position with first coordinate equal to 1. In order to do so, the robber must have moved to the position with first coordinate equal to 0 or 2. He will be caught by the black or red cops dominating G_0 and G_0, respectively. Hence, after the blue cops have dominated G_3, the robber will be at position with the first coordinate not equal to $0,1,2$ and 3.

Suppose that after some time, the black cops are dominating G_0, the red cops are dominating G_k, the blue cops are dominating G_{k+1} and the robber is at position with the first coordinate not equal to $0,1,2,\ldots,k$ and $k+1$. Now, the black and blue cops should remain in place, while the red cops rearrange themselves one by one to dominate G_{k+2}. As before, after the red cops have dominated G_{k+2}, the robber will be at position with the first coordinate not equal to $0,1,2,\ldots,k+1$ and $k+2$. Proceeding in this manner, the robber will be caught when the red cops and blue cops dominated G_{m-1} and G_m. This completes the proof of the theorem. □

Lower bound

Before we proceed to the lower bound, we define the following. Let $0<\alpha<1$ and

$$\rho = \lfloor n^{1-\alpha} \rfloor . \quad (2)$$

Then, $n^{1-\alpha}-1<\rho \leq n^{1-\alpha}$. Here, $\lfloor x \rfloor$ is the floor function, that is the largest integer less than or equal to x.

Let

$$w_i = \begin{cases} \dfrac{n-2}{\binom{n-2}{i}}, & \text{for } 1 \leq i \leq \rho+1; \\ 0, & \text{for } \rho+2 \leq i \leq n. \end{cases} \quad (3)$$

Then, $w_1=1$, $w_2=\frac{2}{n-3}$, $w_3=\frac{6}{(n-3)(n-4)}$.

The following is Stirling's formula, which is quoted from [6, 3.6.2 on p. 31].

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+O\left(\frac{1}{n}\right)\right).$$

Lemma 3.1

$$\binom{n-2}{\rho}^{-1} = O\left(\sqrt{\rho}\left(\frac{\rho}{n}\right)^{\rho}\right).$$

Proof. Note that $\lim_{n\to\infty} \frac{\binom{n}{\rho}}{\binom{n-2}{\rho}} = \lim_{n\to\infty} \frac{n(n-1)}{(n-\rho)(n-\rho-1)} = 1$. So, for sufficiently large n,

$\binom{n-2}{\rho}^{-1} = O\left(\binom{n}{\rho}^{-1}\right)$. By Stirling's formula,

$$\begin{aligned}\binom{n}{\rho}^{-1} &= \frac{(n-\rho)!\,\rho!}{n!}\\ &= O\left(\frac{\left(\sqrt{2\pi(n-\rho)}\left(\frac{n-\rho}{e}\right)^{n-\rho}\right)\left(\sqrt{2\pi(\rho)}\left(\frac{\rho}{e}\right)^{\rho}\right)}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}}\right)\\ &= O\left(\sqrt{\rho}\left(1-\frac{\rho}{n}\right)^{n-\rho}\left(\frac{\rho}{n}\right)^{\rho}\right)\\ &= O\left(\sqrt{\rho}\left(\frac{\rho}{n}\right)^{\rho}\right),\end{aligned}$$

where the last inequality follows from $\left(1-\frac{\rho}{n}\right)^{n-\rho} < 1$. □

Lemma 3.2 *For* $1 \le i \le \rho+1$, w_i *is a strictly decreasing sequence.*

Proof. Since $\frac{n-1-x}{x}$ is a decreasing function, we have

$$\begin{aligned}\frac{w_{i-1}}{w_i} &= \frac{n-1-i}{i}\\ &\ge \frac{n-1-(\rho+1)}{\rho+1} = \frac{n-1}{\rho+1}-1\\ &\ge \frac{n-1}{n^{1-\alpha}+1}-1 = n^{\alpha}\left(\frac{1}{1+\frac{1}{n^{1-\alpha}}}\right)-\frac{1}{n^{1-\alpha}+1}-1 > 1,\end{aligned}$$

for sufficiently large n. □

We are now ready to prove a lower bound on $c_L(\Gamma(m,n))$. Our proof is a generalization of [3, Theorem 1] and [11].

Theorem 3.3 *Let* m,n *be positive integers. Then, for sufficiently large* n *(independent of* m*),*

$c_L(\Gamma(m,n)) \ge (n^{\frac{\alpha}{2}})^{n^{1-\alpha}}$.

Proof. Let N_i be the number of cops that differ at i coordinates from the robber's position. We say that a cop differ at i coordinates from the robber's position has *weight* w_i, where w_i is as defined in equations (3). Let the *potential function* P be defined as

$$P = \sum_{i=1}^{n} N_i w_i.$$

Recall that $w_1 = 1$. If the cops can catch the robber on their turn, then some cop must be at distance 1 from the robber. This implies that the cop differs at exactly 1 coordinate from the robber's position. So, $P \geq 1$, just before the cops' turn. To show that the robber can evade the cops indefinitely, we need to show that the robber can always move right before the cops' move,

$$P < 1. \tag{4}$$

Without loss of generality, all cops start at the same vertex and the robber starts at a vertex with n coordinates different from the cops. Therefore, $P = 0$ and hence (4) holds. Suppose that before the cops make their move, the potential function is such that (4) is satisfied.

Let the coordinate of the robber be $\mathbf{u} = (u_1, \ldots, u_n)$ right before the cops' move. Suppose that on the cops' turn, a cop C^* moves from vertex $\mathbf{v}_0$ to vertex $\mathbf{v}_1$. Suppose $\mathbf{v}_0$ and $\mathbf{u}$ differ at exactly k coordinates. Note that $k \neq 1$ because $P < 1$. So, $2 \leq k \leq n$. If $\mathbf{v}_1$ and $\mathbf{u}$ also differ at k or $k+1$ coordinates, then the robber may remain in $\mathbf{u}$ and the condition (4) is maintained. So, we may assume that $\mathbf{v}_1$ and $\mathbf{u}$ differ at $k-1$ coordinates. We shall choose two distinct integers i_0 and i_1 based on coordinates for which $\mathbf{v}_1$ and $\mathbf{u}$ differ. Let S be the set of all $a \in [n]$ for which the a-th coordinate of $\mathbf{v}_1$ and $\mathbf{u}$ are different. If $|S| = 1$, then choose $i_0 \in S$ and i_1 to be any integer in $[n] Ç S$. If $|S| \geq 2$, then choose $i_0, i_1 \in S$.

Let P_1 represent the total weight of all cops that they differ at i coordinates from $\mathbf{u}$ with $2 \leq i \leq \rho$ other than C^*. Let P_2 represent the total weight of all cops at distance at least $\rho + 1$ from $\mathbf{u}$ other than C^*.

The robber's strategy is to move to a vertex such that the condition (4) is maintained. We will show that such a vertex always exists by computing the expected potential function for all possible vertices the robber can move to. Note that we only allow the robber to move to a vertex that differs with $\mathbf{u}$ at coordinate a where $a \notin \{i_0, i_1\}$. For each $a \notin \{i_0, i_1\}$, the robber can only move to $(u_1, \ldots, u_{a-1}, u_a \pm 1, u_{a+1}, \ldots, u_n)$. Thus, the robber has $2(n-2)$ possible vertices to move to.

We claim that the expected value of P_1 is at most

$$\frac{P_1}{2} \tag{5}$$

after the robber's move in each one round.

Proof of claim (5). Let C and $\mathbf{u}$ differ at coordinates $z_1, \ldots, z_i$ where $2 \leq i \leq \rho$. Before the robber's move, C has weight w_i. Let w_c represent the expected weight of C after the robber's move.

Suppose $i_0, i_1 \in \{z_1, \ldots, z_i\}$. If the robber moves to a vertex $\bar{\mathbf{u}}$ that differs with $\mathbf{u}$ at coordinate a and $a \notin \{z_1, \ldots, z_i\}$, then the number of coordinates that C and $\bar{\mathbf{u}}$ differ is $i+1$. Suppose the robber moves to a vertex that differs with $\mathbf{u}$ at coordinate z_j and $z_j \notin \{i_0, i_1\}$. Note that the robber can only move to $(u_1,\ldots,u_{z_j-1},u_{z_j}\pm 1,u_{z_j+1},\ldots,u_n)$. Let x be the value of the z_j-th coordinate of C.

If $x = u_{z_j}+1$, then the number of coordinates that C and $(u_1,\ldots,u_{z_j-1},u_{z_j}+1,u_{z_j+1},\ldots,u_n)$ differ is $i-1$, and the number of coordinates that C and $(u_1,\ldots,u_{z_j-1},u_{z_j}-1,u_{z_j+1},\ldots,u_n)$ differ is i. In this scenario, one of the choices of the robber will reduce the weight by 1, whereas the other choice will maintain the weight. Similarly, the same conclusion holds, if $x = u_{z_j}-1$. If $x \neq u_{z_j}\pm 1$, then any of the two choices of the robber will maintain the weight. By Lemma 3.2, $w_{i-1} > w_i$, we may assume that for each $z_j \in \{z_1,\ldots,z_i\} \subsetneq \{i_0,i_1\}$, one of the choices of the robber will reduce the weight by 1, whereas the other choice will maintain the weight.
Hence,

$$\begin{aligned} w_C &\le \frac{i-2}{2(n-2)}w_{i-1}+\frac{i-2}{2(n-2)}w_i+\frac{2(n-i)}{2(n-2)}w_{i+1} \\ &\le \frac{i}{2(n-2)}w_{i-1}+\frac{i}{2(n-2)}w_i+\frac{2(n-i-2)}{2(n-2)}w_{i+1}. \end{aligned}$$

Similarly, if $|\{i_0,i_1\}\cap\{z_1,\ldots,z_i\}|=1$, then

$$\begin{aligned} w_C &= \frac{i-1}{2(n-2)}w_{i-1}+\frac{i-1}{2(n-2)}w_i+\frac{2(n-i-1)}{2(n-2)}w_{i+1} \\ &\le \frac{i}{2(n-2)}w_{i-1}+\frac{i}{2(n-2)}w_i+\frac{2(n-i-2)}{2(n-2)}w_{i+1}. \end{aligned}$$

Finally, if $|\{i_0,i_1\}\cap\{z_1,\ldots,z_i\}|=0$, then

$$w_C \le \frac{i}{2(n-2)}w_{i-1}+\frac{i}{2(n-2)}w_i+\frac{2(n-i-2)}{2(n-2)}w_{i+1}.$$

In either case, we have

$$\begin{aligned} w_C &\le \frac{i}{2(n-2)}w_{i-1}+\frac{i}{2(n-2)}w_i+\frac{2(n-i-2)}{2(n-2)}w_{i+1} \\ &= w_i\left(\frac{i}{2(n-2)}\frac{w_{i-1}}{w_i}+\frac{i}{2(n-2)}+\frac{(n-i-2)}{(n-2)}\frac{w_{i+1}}{w_i}\right) \\ &= w_i\left(\frac{i}{2(n-2)}\left(\frac{n-1-i}{i}\right)+\frac{i}{2(n-2)}+\frac{(n-i-2)}{(n-2)}\left(\frac{i+1}{n-2-i}\right)\right) \\ &= w_i\left(\frac{n-1-i}{2(n-2)}+\frac{i}{2(n-2)}+\frac{i+1}{(n-2)}\right) \\ &= w_i\left(\frac{n+1+2i}{2(n-2)}\right) \\ &\le w_i\left(\frac{n-1+2\rho}{2(n-2)}\right) \le \frac{w_i}{2}, \end{aligned}$$

for sufficiently large n as $\lim_{n\to\infty}\frac{\rho}{n}=0$. By summing up each cop's individual contribution toward the potential, we see that after the robber's move, (5) holds. □

Next, we claim that the expected value of P_2 is at most

$$P_2+\frac{1}{n} \tag{6}$$

after the robber's move in each round.

Proof of claim (6). Let C and u differ at i coordinates, where $\rho+1 \le i \le n$. Before the robber's move, C has weight w_i. After the robber's move, the change in the weight of C is either 0, $w_{i+1}-w_i$, or $w_{i-1}-w_i$. Recall that $w_i = 0$ for all $\rho+2 \le i \le n$. If $i \ge \rho+3$, then the change of weight in C is 0. If $i = \rho+2$, then the change of weight in C is either 0, $w_{\rho+3}-w_{\rho+2}=0$, or $w_{\rho+1}-w_{\rho+2}=w_{\rho+1}<w_\rho$ where the last inequality follows from Lemma 3.2. If $i=\rho+1$, then the change of weight in C is either 0, $w_{\rho+2}-w_{\rho+1}=-w_{\rho+1}$, or $w_\rho - w_{\rho+1} < w_\rho$. Hence, the change of weight in C is at most w_ρ.

Since $n^{1-\alpha}-1<\rho\le n^{1-\alpha}$ and by Lemma 3.1, we see that after the robber's move, total change in the weight of C is at most

$$\begin{aligned} w_\rho &= \frac{n-2}{\binom{n-2}{\rho}} = O\left(n\sqrt{\rho}\left(\frac{\rho}{n}\right)^\rho\right) \\ &= O\left(n^{\frac{3-\alpha}{2}}\left(\frac{\rho}{n}\right)^\rho\right) = O\left(n^{\frac{3-\alpha}{2}}\left(\frac{1}{n^\alpha}\right)^\rho\right) \\ &= O\left(\frac{1}{n^{\alpha n^{1-\alpha}-\frac{3+\alpha}{2}}}\right). \end{aligned}$$

If the total number of cops is at most $(n^{\frac{\alpha}{2}})^{n^{1-\alpha}}$, then the expected change in P_2 is at most

$$O\left(\frac{n^{\frac{\alpha n^{1-\alpha}}{2}}}{n^{\alpha n^{1-\alpha}-\frac{3+\alpha}{2}}}\right) = O\left(\frac{1}{n^{\frac{\alpha n^{1-\alpha}}{2}-\frac{3+\alpha}{2}}}\right),$$

and (6) follows. □

Recall that on the cops' turn, a cop C^* moves from vertex $\mathbf{v}_0$ to vertex $\mathbf{v}_1$. We shall assume that $\mathbf{v}_0$ and $\mathbf{u}$ differ at exactly k coordinates, whereas $\mathbf{v}_1$ and $\mathbf{u}$ differ at exactly $k-1$ coordinates.

Case 1. Suppose $k=2$. Recall that the robber is allowed to move to a vertex that differs with $\mathbf{u}$ at coordinate a where $a \notin \{i_0, i_1\}$. So, after the robber's move, (5) and (6) hold.

Since C^* has weight w_2 before its move, we have

$$P_1 + P_2 + w_2 < 1. \tag{7}$$

After the robber's move, C^* also has weight w_2. Combining (5), (6) and (7), the total expected potential is at most

$$\begin{aligned} &\frac{P_1}{2} + w_2 + P_2 + \frac{1}{n} \\ &< \frac{1-w_2-P_2}{2} + w_2 + P_2 + \frac{1}{n} \\ &= \frac{1}{2} + \frac{w_2+P_2}{2} + \frac{1}{n} \\ &< 1, \end{aligned}$$

where the second to last inequality follows from $w_2 = O\left(\frac{1}{n}\right)$ and $P_2 = O\left((n^{\frac{\alpha}{2}})^{n^{1-\alpha}} w_\rho\right) = O\left(\frac{1}{n}\right)$.

Since the expected potential is less than 1, there must be a move for the robber such that (4) is maintained.

Case 2. Suppose $k=3$. Then, $\mathbf{v}_1$ and $\mathbf{u}$ differ at exactly 2 coordinates. In this scenario, C^* moves from a vertex of weight w_3 to another vertex of weight w_2. We note here that there is the possibility that other cops are at vertex with weight w_2. Recall that the robber is allowed to move to a vertex that differs with $\mathbf{u}$ at coordinate a where $a \notin \{i_0, i_1\}$. So, after the robber's move, (5) and (6) hold.

Since C^* has weight w_3 before its move, we have

$$P_1 + P_2 + w_3 < 1. \tag{8}$$

After the robber's move, C^* also has weight w_3. Combining (5), (6) and (8), the total expected potential is at most

$$\begin{aligned}
&\frac{P_1}{2} + w_3 + P_2 + \frac{1}{n} \\
&< \frac{1 - w_3 - P_2}{2} + w_3 + P_2 + \frac{1}{n} \\
&= \frac{1}{2} + \frac{w_3 + P_2}{2} + \frac{1}{n} \\
&< 1,
\end{aligned}$$

where the second to last inequality follows from $w_3 = O\left(\frac{1}{n^2}\right)$ and $P_2 = O\left(\frac{1}{n}\right)$. Again, the robber may maintain (4) is satisfied.

Case 3. Suppose $k \geq 4$. Then, $\mathbf{v}_1$ and $\mathbf{u}$ differ at exactly $k-1 \geq 3$ coordinates. Again, like the previous cases, after the robber's move, (5) and (6) hold.
Since C^* has weight w_k before its move, we have $P_1 + P_2 + w_k < 1$. Thus,

$$P_1 + P_2 < 1 - w_k < 1. \tag{9}$$

Now, C^* has weight w_{k-1} just before the robber's move. Let w_{C^*} represent the expected weight of C^* after the robber's move. If $3 \leq k-1 \leq \rho$, then $w_{C^*} \leq \frac{w_{k-1}}{2} = O(w_3) = O\left(\frac{1}{n^2}\right)$. If $\rho + 1 \leq k-1 \leq n-1$, then $w_{C^*} \leq w_{k-1} + O\left(\frac{1}{n}\right) = O(w_3) + O\left(\frac{1}{n}\right) = O\left(\frac{1}{n}\right)$.
Hence, the expected potential is at most

$$\begin{aligned}
&\frac{P_1}{2} + P_2 + \frac{1}{n} + w_{C^*} \\
&< \frac{1 - P_2}{2} + P_2 + {+}O\left(\frac{1}{n}\right) \\
&= \frac{1}{2} + \frac{P_2}{2} + \frac{1}{n} + O\left(\frac{1}{n}\right) \\
&< 1,
\end{aligned}$$

where the last inequality follows from $P_2 = O\left(\frac{1}{n}\right)$.
This completes the proof of the theorem. □

Combining Theorem 2.2 and Theorem 3.3, we obtain our main result as given below.

Main Results

Theorem 4.1 *Let* m *be a positive integer and let* $0 < \alpha < 1$. *Then for sufficiently large* n *, we have*

$$(m+1)^{n^{1-\alpha}} \leq c_L(\Gamma(m,n)) \leq \frac{3(m+1)^{n-1}\ln n}{n}.$$

Proof. The upper bound follows readily from Theorem 2.2, while the lower bound follows from Theorem 3.3 by taking $n^{\frac{\alpha}{2}} > (m+1)$. □

Data availability

A data availability statement is compulsory for research articles and clinical trials. Here, authors must describe how readers can access the data underlying the findings of the study, giving links to online repositories and providing deposition codes where applicable.

Conflicts of interest

The author(s) declare(s) that there is no conflict of interest regarding the publication of this paper.

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