MJFAS
Malaysian Journal of Fundamental and Applied Sciences

RESEARCH ARTICLE

Coefficient Estimates of Toeplitz Determinant for a Certain Class of Close-to-Convex Functions

Shaharuddin Cik Soh, Daud Mohamad, Huzaifah Dzubaidi*

Department of Mathematics, Faculty of Computer and Mathematical Sciences, Universiti Teknologi Mara (UITM), 40450 Shah Alam, Selangor Darul Ehsan, Malaysia

Abstract Let S denote the class of analytic and univalent functions in D, where D is defined as unit disk, $D \coloneqq \{z \in C : |z| < 1\}$ and having the Taylor representation form of S. We will determine the estimation for the Toeplitz determinants where the elements are the Taylor coefficients of the class close-to-convex functions in S.

Keywords: Univalent functions, Analytic functions, Toeplitz determinants, Close-to-convex functions

***For correspondence:**
2018610662@isiswa.uitm.edu.my

Received: 15 January 2021
Accepted: 15 October 2021

Introduction and Preliminaries

Let A denote the class of analytic functions normalized by $f(0) = 0 = f'(0) - 1$ in D, where D is defined as the unit disk, $D \coloneqq \{z \in C : |z| < 1\}$. If $f \in A$ then $f(z)$ has the following representation.

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

We denote S as the class of univalent functions in A. There are three major subclasses in S that include starlike functions, convex functions, and close-to-convex functions. Let S^* represent the class of starlike functions in S. A function $f \in A$ becomes a starlike function, if it satisfies the following condition,

$$\mathrm{Re}\left(\frac{z f'(z)}{f(z)}\right) > 0 \tag{2}$$

for $z \in D$. In the starlike functions, it has an important member as well, as the class S is the Koebe function that is defined as follows,

$$k(z) = \frac{z}{(1-z)^2} \tag{3}$$

In most issues for the S^* and S classes, the Koebe function play the role of extreme function. A function $f \in A$ is supposed to be a convex function, if it meets the following condition,

$$\mathrm{Re}\left(1 + \frac{z f''(z)}{f'(z)}\right) > 0 \tag{4}$$

for $z \in D$. We denote CV for the class of convex functions in S. In 1952, Kaplan had stated that if function $f \in S$ and if there exists a real number, α, where $|\alpha| < \pi/2$ and a function $g(z)$ is convex which satisfy these conditions,

$$\mathrm{Re}\left(e^{i\alpha} \frac{f'(z)}{g'(z)} \right) > 0 \tag{5}$$

where $z \in D$ [14]. Then the function f is a close-to-convex function. Alexander in 1916 showed the connection between starlike and convex functions if function $h(z) \in S^*$ then $h(z) = zg'(z)$ where $g(z) \in CV$ [10]. Hence, the condition (5) can also be written as

$$\mathrm{Re}\left(e^{i\alpha} \frac{zf'(z)}{h(z)} \right) > 0 \tag{6}$$

where $z \in D$. For all starlike and convex functions are close-to-convex functions. These can be summarized by proper inclusion $CV \subseteq S^* \subseteq K \subseteq S$. The class of close-to-convex functions is denoted by K. From there, we are inclined to define a class of close-to-convex functions with new additional parameters and find the coefficient estimates of Toeplitz determinants for the class defined.

Definition 1. The function $f \in A$ is said to be a close-to-convex if the starlike function $g \in S^*$ exist, as such that

$$\mathrm{Re}\left(e^{i\alpha} \frac{z f'(z)}{g(z)} \right) > \delta \tag{7}$$

where $z \in D$, $|\alpha| < \pi/2$, and $\cos(\alpha) > \delta$. This class is denoted by $K_{\alpha\delta}$.

In many branches of mathematics, the Hankel matrices and determinants played significant role and have several applications. There are several results of the Hankel determinants in the literature. The early investigation of Hankel determinants for many classes of analytic functions started in the 1960s. In 1966 and 1967, Pommerenke had studied Hankel's determinants for the class of univalent functions given by [6] (cited by [16]). After that, many recent papers have been concerned to the problem of finding the estimate of Hankel determinants for various subclasses of univalent functions.

Each of them will have to deal with finding the upper bound for $|a_2a_4 - a_3^2|$ of their own classes. Some of the results, will also find the more general functional $|a_2a_4 - \mu a_3^2|$ with the real μ for various classes of functions. For example, the results of [2], [3], [9], [12], [13], [17] and [18].

Closely connected to the Hankel determinants are the Toeplitz determinants. Along with the reverse diagonal, the Hankel matrices have constant entries, while the Toeplitz matrices have constant entries along with the diagonal. We referred [22] for a description of the application of the Toeplitz matrices to a wide variety of areas especially in pure and applied mathematics. The symmetric Toeplitz determinant $T_q(n)$ has been recently introduced by [7] for the analytic functions f of the form (1), defined as follows

$$T_q(n) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_n \end{vmatrix}$$

where $n, q = 1,2,3,\cdots$ and $a_1 = 1$. In particular,

$$T_2(2) := \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix}, \quad T_2(3) := \begin{vmatrix} a_3 & a_4 \\ a_4 & a_3 \end{vmatrix}, \quad T_3(1) := \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & 1 & a_2 \\ a_3 & a_2 & 1 \end{vmatrix}, \quad T_3(2) := \begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_2 & a_3 \\ a_4 & a_3 & a_2 \end{vmatrix}.$$

The estimates of the Toeplitz determinant $|T_q(n)|$ for functions in the starlike and the close-to-convex functions were studied by [7] for small values of n and p. Similarly, for the authors in [19] and [20], they had studied the estimates of the Toeplitz determinants $|T_q(n)|$ for functions belonged to a certain conic domain and bounded rotation, R respectively. There seems to be little in the literature concerning the estimates of the Toeplitz determinants, aside from the result of [7]. The authors in [1] pointed out that the result obtained by [7] is not valid because of the $c_1 > 0$ is not justified, since the functional $|T_q(n)|$ for $n, q = 1,2,3,\cdots$ is not rotationally invariant. Consequently, the authors in [7] agreed to retract their research paper due to the results were not fully proved.

In the following year, the authors in [1] came out with the sharp estimate for the Toeplitz determinants in which the elements were the Taylor coefficients of functions in S and some of its classes. In this purpose paper to give the estimates for Toeplitz determinants $T_q(n)$ for the class $K_{\alpha\delta}$ which is the class of close-to-convex functions. Let P represent the class of analytic functions p in the unit disk D, which has the following form

$$p(z)=1+\sum_{n=1}^{\infty}c_n z^n \tag{8}$$

such that $Re\, p(z)>0$ in the unit disk, D. These functions are sometimes called the Caratheodory functions. Before we continue to prove the main result, we need some preliminary results for the functions in P.

Lemma 2. [11, page 41]. *For a function $p\in P$ of the form (8), the sharp inequality $|c_n|\le 2$ holds for each $n\ge 1$. Equality holds for the function $p(z)=(1+z)/(1-z)$.*

Lemma 3. [8, Theorem 1]. *Let $p\in P$ be the form (8) and $\mu\in C$. Then*

$$|c_n-\mu c_k c_{n-k}|\le 2\,max\{1,|2\mu-1|\} \text{ for } 1\le k\le n-1.$$

If $|2\mu-1|\ge 1$ then the inequality is sharp for the function $p(z)=(1+z)/(1-z)$ or its rotations.
If $|2\mu-1|\le 1$ then the inequality is sharp for $p(z)=(1+z^n)/(1-z^n)$ or its rotations.

Lemma 4. [6, Theorem 1]. *Let $g\in S^*$ be in the form*

$$g(z)=z+\sum_{n=2}^{\infty}b_n z^n \tag{9}$$

Then, for any $\lambda\in C$,

$$\left|b_3-\lambda b_2^2\right|\le \max\{1,|3-4\lambda|\}$$

The inequality is sharp for Koebe function in the form (3) and its rotations if $|3-4\lambda|\ge 1$, and for $(k(z^2))^2$ and its rotations if $|3-4\lambda|<1$.

Lemma 5. [15, Theorem 2.2]. *Let $g\in S^*$ be in the form of (9). Then*

$$\left|\lambda b_n b_m-b_{n+m-1}\right|\le \lambda nm-(n+m-1) \text{ for } \lambda\ge\frac{2(n+m-1)}{nm},$$

where $n,m=2,3,\cdots$. the inequality is sharp for the Koebe function in the form (3) and its rotations.

Lemma 6. [2, Theorem 3.1]. *Let $g\in S^*$ be in the form of (9). Then*

$$\left|b_2 b_4-b_3^2\right|\le 1,$$

and the inequality is sharp for the Koebe function, in the form (3) and its rotations.

Lemma 7. *Let $f\in K_{\alpha\delta}$ be the form (1). Then*

$$\left|a_2a_4-2a_3^2\right|\le\frac{1}{72}\left[340+148\left(A_{\alpha\delta}\right)^2+120A_{\alpha\delta}+36A_{\alpha\delta}\left(\sqrt{\frac{1369}{81}+\frac{2116}{81}\delta^2}\right)\right].$$

Proof: Noted that,

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n=z+a_2z^2+a_3z^3+a_4z^4+\cdots \tag{10}$$

$$f'(z)=1+\sum_{n=2}^{\infty}na_n z^{n-1}=1+2a_2z+3a_3z^2+4a_4z^3+\cdots \tag{11}$$

$$z f'(z) = z + \sum_{n=2}^{\infty} n a_n z^n = z + 2a_2 z^2 + 3a_3 z^3 + 4a_4 z^4 + \cdots \quad (12)$$

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n = z + b_2 z^2 + b_3 z^3 + b_4 z^4 + \cdots \quad (13)$$

and

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \quad (14)$$

By referring to the definition 1, we have

$e^{i\alpha} \dfrac{zf'(z)}{g(z)} > \delta$, where $\cos(\alpha) > \delta, |\alpha| < \dfrac{\pi}{2}$.

A Caratheodory function exists, as $p \in P$ of the form (14) as such that,

$$e^{i\alpha} \frac{zf'(z)}{g(z)} - \delta = p(z). \quad (15)$$

Firstly, we need to find the representation theorem of class $K_{\alpha\delta}$. Let $f \in S$ be in the form of (1), then from Left-Hand Side (LHS) of the equation (15), it can be written in the form,

$$e^{i\alpha} \frac{zf'(z)}{g(z)} - \delta = e^{i\alpha}\left(1 + \sum_{n=1}^{\infty} c_n z^n\right) - \delta,$$

and

$$e^{i\alpha} \frac{zf'(z)}{g(z)} - \delta - i\sin(\alpha) = \cos(\alpha) - \delta + \sum_{n=1}^{\infty} e^{i\alpha} c_n z^n,$$

which gives

$$\frac{e^{i\alpha} \frac{zf'(z)}{g(z)} - \delta - i\sin(\alpha)}{\cos(\alpha) - \delta} = 1 + \sum_{n=1}^{\infty} \left(\frac{e^{i\alpha} c_n}{\cos(\alpha) - \delta}\right) z^n,$$

and

$$\frac{e^{i\alpha} \frac{zf'(z)}{g(z)} - \delta - i\sin(\alpha)}{\cos(\alpha) - \delta} = 1 + \sum_{n=1}^{\infty} q_n z^n , \quad (16)$$

where $q_n = \left(\dfrac{e^{i\alpha} c_n}{\cos(\alpha) - \delta}\right)$. Then, we can relate the equation (16) to the function in P with

$$\frac{e^{i\alpha} \frac{zf'(z)}{g(z)} - \delta - i\sin(\alpha)}{\cos(\alpha) - \delta} = p(z),$$

where $p(z)$ has the form of (8). Take note that, $\cos(\alpha) - \delta$ must always be positive. Next, we find the coefficients of z, by using the following equation,

$$e^{i\alpha} \frac{zf'(z)}{g(z)} - \delta - i\sin(\alpha) = p(z)(\cos(\alpha) - \delta)$$

which implies,

$$zf'(z) = \left(e^{-i\alpha}\right)[g(z)][p(z)(\cos(\alpha) - \delta) + \delta + i\sin(\alpha)] \quad (17)$$

Take note that, $A_{\alpha\delta} = \cos(\alpha) - \delta$ and from the Right-Hand Side (RHS) of the equation (17), we have

$$\left(e^{-i\alpha}\right)\left(z+\sum_{n=2}^{\infty}b_nz^n\right)\left[\left(1+\sum_{n=1}^{\infty}c_nz^n\right)(\cos(\alpha)-\delta)+\delta+i\sin(\alpha)\right]$$

$$=\left(e^{-i\alpha}\right)\left(z+\sum_{n=2}^{\infty}b_nz^n\right)\left[\left(\cos(\alpha)-\delta+\sum_{n=1}^{\infty}(\cos(\alpha)-\delta)c_nz^n\right)+\delta+i\sin(\alpha)\right]$$

$$=\left(e^{-i\alpha}\right)\left(z+\sum_{n=2}^{\infty}b_nz^n\right)\left(e^{i\alpha}+\sum_{n=1}^{\infty}A_{\alpha\delta}\,c_nz^n\right)$$

Also, from the Left-Hand Side (LHS) of the equation (17), we have

$$zf'(z)=z+\sum_{n=2}^{\infty}na_nz^n=z+2a_2z^2+3a_3z^3+4a_4z^4+\cdots$$

By comparing the coefficients of z in both LHS and RHS of the equation (17), we will get

$$2a_2=A_{\alpha\delta}\,e^{-i\alpha}c_1+b_2 \tag{18}$$

$$3a_3=A_{\alpha\delta}\,e^{-i\alpha}c_2+A_{\alpha\delta}\,e^{-i\alpha}\,b_2c_1+b_3 \tag{19}$$

$$4a_4=A_{\alpha\delta}\,e^{-i\alpha}c_3+A_{\alpha\delta}\,e^{-i\alpha}\,b_2c_2+A_{\alpha\delta}\,e^{-i\alpha}\,b_3c_1+b_4 \tag{20}$$

Now, we must find the equation $a_2a_4-2a_3^2$ by using the coefficients that we had obtained in (18), (19) and (20). We let $A=a_2a_4$,

$$A=\left(\frac{1}{2}\right)\left(A_{\alpha\delta}\,e^{-i\alpha}c_1+b_2\right)\left(\frac{1}{4}\right)\left(A_{\alpha\delta}\,e^{-i\alpha}c_3+A_{\alpha\delta}\,e^{-i\alpha}\,b_2c_2+A_{\alpha\delta}\,e^{-i\alpha}\,b_3c_1+b_4\right)$$

$$=\left(\frac{1}{8}\right)\left[\left(A_{\alpha\delta}\right)^2e^{-2i\alpha}\left(c_1c_3+b_2c_2c_1+b_3c_1^2\right)+A_{\alpha\delta}\,e^{-i\alpha}\left(b_4c_1+b_2c_3+b_2^2c_2+b_3b_2c_1\right)+b_4b_2\right]$$

Also, by letting $B=2a_3^2$, we will have

$$B=\left(\frac{2}{9}\right)\left(A_{\alpha\delta}\,e^{-i\alpha}c_2+A_{\alpha\delta}\,e^{-i\alpha}\,b_2c_1+b_3\right)^2$$

$$=\left(\frac{2}{9}\right)\left[\left(A_{\alpha\delta}\right)^2e^{-2i\alpha}\left(c_2^2+2b_2c_2c_1+b_2^2c_1^2\right)+A_{\alpha\delta}\,e^{-i\alpha}\left(2b_3b_2c_1+2b_3c_2\right)+b_3^2\right]$$

Then, we will subtract the equations of A and B that we have obtained so that it becomes $A-B=a_2a_4-2a_3^2$, and it yields

$$A-B=\left[\left(A_{\alpha\delta}\right)^2e^{-2i\alpha}\left(\frac{c_1c_3}{8}+\frac{b_2c_2c_1}{8}+\frac{b_3c_1^2}{8}-\frac{2c_2^2}{9}-\frac{4b_2c_2c_1}{9}-\frac{2b_2^2c_1^2}{9}\right)\right.$$
$$\left.+A_{\alpha\delta}\,e^{-i\alpha}\left(\frac{b_4c_1}{8}+\frac{b_2c_3}{8}+\frac{b_2^2c_2}{8}+\frac{b_3b_2c_1}{8}-\frac{4b_3b_2c_1}{9}-\frac{4b_3c_2}{9}\right)+\frac{b_4b_2}{8}-\frac{2b_3^2}{9}\right]$$

and when simplify it further, we get

$$\begin{aligned}A-B=\frac{1}{72}&\left[\left(9b_4b_2-16b_3^2\right)+A_{\alpha\delta}\,e^{-i\alpha}c_1\left(9b_4-23b_3b_2\right)\right.\\&+\left(A_{\alpha\delta}\right)^2e^{-2i\alpha}c_1^2\left(9b_3-16b_2^2\right)+A_{\alpha\delta}\,e^{-i\alpha}c_2\left(9b_2^2-32b_3\right)\\&\left.+A_{\alpha\delta}\,e^{-i\alpha}b_2\left(9c_3-23A_{\alpha\delta}\,e^{-i\alpha}c_2c_1\right)+\left(A_{\alpha\delta}\right)^2e^{-2i\alpha}\left(9c_1c_3-16c_2^2\right)\right]\end{aligned} \tag{21}$$

By applying the triangle inequality for the equation (21), we have

$$\begin{aligned}72\left|a_2a_4-2a_3^2\right|\le&\left|9b_4b_2-16b_3^2\right|+A_{\alpha\delta}\left|c_1\right|\left|9b_4-23b_3b_2\right|\\&+\left(A_{\alpha\delta}\right)^2\left|c_1^2\right|\left|9b_3-16b_2^2\right|+A_{\alpha\delta}\left|c_2\right|\left|9b_2^2-32b_3\right|\\&+A_{\alpha\delta}\left|b_2\right|\left|9c_3-23A_{\alpha\delta}\,e^{-i\alpha}c_2c_1\right|+\left(A_{\alpha\delta}\right)^2\left|9c_1c_3-16c_2^2\right|\end{aligned} \tag{22}$$

By letting $Y_1=|9b_4b_2-16b_3^2|$, $Y_2=|9b_4-23b_3b_2|$, $Y_3=|9b_3-16b_2^2|$, $Y_4=|9b_2^2-32b_3|$,

$Y_5 = |9c_3 - 23A_{\alpha\delta}e^{-i\alpha}c_2c_1|$, and $Y_6 = |9c_1c_3 - 16c_2^2|$. Then, by applying the Lemma 4, 5, and 6, we will get

$$\begin{aligned} Y_1 &\le 9\left|b_4b_2 - b_3^2\right| + 7\left|b_3\right|^2 \\ &\le 9 + 63 \\ &\le 72 \end{aligned} \tag{23}$$

$$\begin{aligned} Y_2 &\le 9\left|b_4 - \frac{23b_3b_2}{9}\right| \\ &\le 9\left(\frac{46}{3} - 4\right) \\ &\le 102 \end{aligned} \tag{24}$$

$$\begin{aligned} Y_3 &\le 9\left|b_3 - \frac{16b_2^2}{9}\right| \\ &\le 9\left(\frac{64}{9} - 3\right) \\ &\le 37 \end{aligned} \tag{25}$$

$$\begin{aligned} Y_4 &\le 32\left|b_3 - \frac{9b_2^2}{32}\right| \\ &\le 32\left(3 - \frac{9}{8}\right) \\ &\le 60 \end{aligned} \tag{26}$$

For $Y_5 = 9|c_3 - \mu c_2c_1| \le 18\max\{1, |2\mu - 1|\}$, where $\mu = \frac{23}{9}A_{\alpha\delta}e^{-i\alpha}$.Take note that,

$$\begin{aligned} |2\mu - 1|^2 &= \left|2\left(\frac{23}{9}A_{\alpha\delta}e^{-i\alpha}\right) - 1\right|^2 \\ &= \left[\frac{14}{9} + \frac{23}{9}\cos(2\alpha) - \frac{46}{9}\delta\cos(\alpha)\right]^2 + \left[\frac{23}{9}\sin(2\alpha) - \frac{46}{9}\delta\sin(\alpha)\right]^2. \end{aligned} \tag{27}$$

Firstly, we need to expand the equation (27) by letting

$Q_1 = \left[\frac{14}{9} + \frac{23}{9}\cos(2\alpha) - \frac{46}{9}\delta\cos(\alpha)\right]^2$ and $Q_2 = \left[\frac{23}{9}\sin(2\alpha) - \frac{46}{9}\delta\sin(\alpha)\right]^2$.

From Q_1 we will have,

$$Q_1 = \frac{196}{81} + \frac{644}{81}\cos(2\alpha) + \frac{529}{81}(\cos(2\alpha))^2 + \frac{2116}{81}\delta^2(\cos(\alpha))^2.$$

Then from Q_2 we will have,

$$Q_2 = \frac{529}{81}(\sin(2\alpha))^2 - \frac{2116}{81}\delta\sin(2\alpha)\sin(\alpha) + \frac{2116}{81}\delta^2(\sin(\alpha))^2.$$

Hence, we will add both equations, Q_1 and Q_2, so we will get

$$Q_1 + Q_2 = \frac{725}{81} + \frac{644}{81}\cos(2\alpha) + \frac{2116}{81}\delta^2 - \frac{2116}{81}\delta\sin(2\alpha)\sin(\alpha).$$

By letting $f(\alpha) = Q_1 + Q_2$, then we need to find the critical point of the equation by differentiating the function $f(\alpha)$ with respect to α, so we will have

$$\begin{aligned} \frac{df(\alpha)}{d\alpha} &= \frac{1288}{81}\sin(2\alpha) - \frac{2116}{81}\delta(\sin(2\alpha)\cos(\alpha) + 2\sin(\alpha)\cos(2\alpha)) \\ &= \frac{1}{81}\left[\sin(2\alpha)(1288 - 2116\delta(\cos(\alpha) + 2\sin(\alpha)\cot(2\alpha)))\right]. \end{aligned} \tag{28}$$

After differentiating the function $f(\alpha)$, we will let the differential equation (28) to zero, and we will get

$$\frac{1}{81}\left[\sin(2\alpha)(1288-2116\delta(\cos(\alpha)+2\sin(\alpha)\cot(2\alpha)))\right]=0.$$

Since the equation $\sin(2\alpha)$ can only be equivalent to zero and we had obtained the following critical points,

$\alpha=\frac{\pi}{2}$ and $\alpha=0$.

From the critical points that we had acquired, we know that,

$$1+\frac{2116\delta^2}{81}\leq|2\mu-1|^2\leq\frac{1369}{81}+\frac{2116}{81}\delta^2$$

and

$$\sqrt{1+\frac{2116\delta^2}{81}}\leq|2\mu-1|\leq\sqrt{\frac{1369}{81}+\frac{2116}{81}\delta^2}$$

Therefore,

$$Y_5=\left|9c_3-23A_{\alpha\delta}e^{-i\alpha}c_2c_1\right|\leq 18\left(\sqrt{\frac{1369}{81}+\frac{2116}{81}\delta^2}\right) \quad (29)$$

Again, by using Lemma 2 and 3 for the equation Y_6, we will have

$$\begin{aligned}\left|9c_1c_3-16c_2^2\right|&\leq 9|c_1c_3-c_4|+9\left|c_4-\frac{16}{9}c_2^2\right|\\&\leq 18+46 \quad (30)\\&\leq 64.\end{aligned}$$

Hence, by using the relations of the equations (23), (24), (25), (26), (29) and (30), the inequality (22), we will get

$$\begin{aligned}\left|a_2a_4-2a_3^2\right|\leq\frac{1}{72}&\Big[72+(A_{\alpha\delta})(2)(102)+(A_{\alpha\delta})^2(2)^2(37)+A_{\alpha\delta}(2)(60)\\&+A_{\alpha\delta}(2)(18)\left(\sqrt{\frac{1369}{81}+\frac{2116}{81}\delta^2}\right)+64\Big]\\&\leq\frac{1}{72}\left[340+148(A_{\alpha\delta})^2+120A_{\alpha\delta}+36A_{\alpha\delta}\left(\sqrt{\frac{1369}{81}+\frac{2116}{81}\delta^2}\right)\right].\end{aligned}$$

This concludes the proof.

Theorem 2.9. *Let $f\in K_{\alpha\delta}$ be in the form of (1). Then*

$$|T_3(2)|\leq 44+4\left[\frac{340}{72}+\frac{148}{72}(A_{\alpha\delta})^2+\frac{120}{72}A_{\alpha\delta}+\frac{36}{72}A_{\alpha\delta}\left(\sqrt{\frac{1369}{81}+\frac{2116}{81}\delta^2}\right)\right]$$

Proof. Let $f\in K_{\alpha\delta}$ be in the form of (1). Then by Lemma 7,

$$\begin{aligned}|T_3(2)|&=\left|a_2^3-2a_2a_3^2-a_2a_4^2+2a_3^2a_4\right|\\&\leq|a_2|^3+2|a_2|\left|a_3^2\right|+|a_4|\left|a_2a_4-2a_3^2\right|\\&\leq 44+4\left[\frac{340}{72}+\frac{148}{72}(A_{\alpha\delta})^2+\frac{120}{72}A_{\alpha\delta}+\frac{36}{72}A_{\alpha\delta}\left(\sqrt{\frac{1369}{81}+\frac{2116}{81}\delta^2}\right)\right].\end{aligned}$$

This concludes the proof. By letting $\delta=0$ and $\alpha=0$, the results that were obtained can be reduced to the result of [1].

Acknowledgments

The authors would like to thank the anonymous reviewers for their insightful suggestions and critical reading of the manuscript.

References

[1] Ali, M. F., Thomas, D. K., & Vasudevarao, A. (2018). Toeplitz determinants whose elements are the coefficients of analytic and univalent functions. *Bull. Aust. Math. Soc.*, 97(2), 253-264.

[2] A. Janteng, S. A. Halim and M. Darus, 'Hankel determinant for starlike and convex functions'*, Int. J. Math. Anal*. 1 (2007), 619–625.

[3] Aini Janteng, Suzeini Abdul Halim and Maslina Darus, 2008. Estimate on the second Hankel functional for functions whose derivative has a positive real part. *Journal of Quality Measurement and Analysis*, Vol. 4, pp. 189-195.

[4] Aini Janteng, Suzeini Abdul Halim and Maslina Darus, 2007. Hankel determinant for starlike and convex functions, *International Journal of Mathematical Analysis*, Vol. 1, no. 13, pp. 619-625.

[5] Ali, M. F., & Vasudevarao, A. (2017). A note on Toeplitz Determinants whose elements are the coefficients of Univalent Functions. *arXiv preprint arXiv*:1704.00657.

[6] Bansal, D. (2013). Upper bound of second Hankel determinant for a new class of analytic functions. *Applied Mathematics Letters*, *26*(1), 103-107.

[7] D. K. Thomas and S. A. Halim, RETRACTED ARTICLE: 'Toeplitz matrices whose elements are the coefficients of starlike and close-to-convex functions', *Bull. Malays. Math. Sci. Soc*. 40(4) (2017), 1781–1790.

[8] F. R. Keogh and E. P. Merkes, 'A coefficient inequality for certain classes of analytic functions', *Proc. Amer. Math. Soc*. 20 (1969), 8–12.

[9] Gangadharan, Murugusundaramoorthy and Nanjundan Magesh, 2009. Coefficient inequalities for certain classes of analytic functions associated with Hankel determinant. *Bulletin of Mathematical Analysis and Applications*, Vol. 1, issue 5, pp. 85-89.

[10] Goodman, A. W. (1983). *Univalent functions* (Vol. 1): Mancorp Pub.

[11] I. Efraimidis, 'A generalization of Livingston's coefficient inequalities for functions with positive real part', *J. Math. Anal. Appl*. 435(1) (2016), 369–379.

[12] Oqlah Al-Refai and Maslina Darus, 2009. Second Hankel determinant for a class of analytic functions defined by a Fractional Operator. *European Journal of Scientific Research*, Vol. 28, no. 2, pp. 234-241.

[13] Kharudin, N., Akbarally, A., Mohamad, D., & Soh, S. C. (2011). The second Hankel determinant for the class of close to convex functions. European Journal of Scientific Research, 66(3), 421-427.

[14] Kaplan, W. (1952). Close-to-convex schlicht functions. *The Michigan Mathematical Journal*, 1(2), 169-185.

[15] P. L. Duren, *Univalent Functions*, Grundlehren der mathematischen Wissenschaften, 259 (Springer, New York–Berlin–Heidelberg–Tokyo, 1983).

[16] Răducanu, D., & Zaprawa, P. (2017). Second Hankel determinant for close-to-convex functions. *Comptes Rendus Mathematique*, 355(10), 1063-1071.

[17] Shaharuddin Cik Soh and Daud Mohamad, 2008. Coefficient bounds for certain classes of close-to-convex functions. *International Journal of Mathematical Analysis*, Vol. 2, no. 27, pp. 1343-1351.

[18] Selvaraj, C. and Vasanthi, N., 2010. Coefficient bounds for certain subclasses of close-to convex functions. *International Journal of Mathematical Analysis*, Vol. 4, no. 37, pp. 1807-1814.

[19] Sivasubramanian, S., Govindaraj, M., & Murugusundaramoorthy, G. (2016). Toeplitz Matrices whose Elements are the Coefficients of Analytic Functions belonging to Certain Conic Domains. *International Journal of Pure and Applied Mathematics*, *109*(10), 39-49.

[20] V. Radhika, S. Sivasubramanian, G. Murugusundaramoorthy and J. M. Jahangiri, 'Toeplitz matrices whose elements are the coefficients of functions with bounded boundary rotation', *J. Complex Anal*. 2016 (2016), 4960704.

[21] W. Ma, 'Generalized Zalcman conjecture for starlike and typically real functions', *J. Math. Anal. Appl*. 234(1) (1999), 328–339.

[22] Ye, K., & Lim, L. H. (2016). Every matrix is a product of Toeplitz matrices. *Foundations of Computational Mathematics*, 16(3), 577-598.