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Approximate solution of a nonlinear system of integral equations using modified Newton-Kantorovich method

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ABSTRACT

Modified Newton-Kantorovich method is developed to obtain an approximate solution for a system of nonlinear integral equations. The system of nonlinear integral equations is reduced to find the roots of nonlinear integral operator. This nonlinear integral operator is solved by the modified Newton-Kantorovich method with initial conditions and this procedure is continued by iteration method to find the unknown functions. The existence and uniqueness of the solutions of the system are also proven.

| Newton-Kantorovich method | Nonlinear operator | Volterra integral equation | Trapezoidal rule | Convergence|

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1. INTRODUCTION

Solving the nonlinear integral equations is an important issue in the engineering and technology fields. Determining the roots of an equation has attracted the attention of pure and applied mathematicians for many years ([2, 4, 5]). Many problems may be formulated in terms of finding zeros. These roots cannot in general be expressed in a closed form. Thus, in order to solve nonlinear equations, we have to use approximate methods [1], and the necessity of their approximate solutions was emphasized in [3]. One of the approaches for solving nonlinear integral equations is Newton-Kantorovich method. In 1939, Kantorovich [6] published a paper on iterative methods for functional equations in a Banach space and applied this theory to derive a convergence theorem for Newton's method. Later, in 1948, he [7] established a semilocal convergence theorem for Newton's method in a Banach space, which is now called Kantorovich's theorem or the Newton-Kantorovich theorem. Newton's method which is defined by

$$x_{n+1} = x_n - [P'(x_n)]^{-1} P(x_n) \qquad (n \ge 0)(x_0 \in \Omega) \quad (1)$$

has been used extensively by many authors to generate a sequence $\{x_n\}(n \ge 0)$ converging to x^*. This method was proposed by Newton in 1669, for finding the roots of polynomials $P_n(x)$.

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2. DESCRIPTION OF THE METHOD

In this paper we consider the adaptation of [4] for the system of nonlinear integral equations

$$\begin{cases} x(t) - \displaystyle\int_{y(t)}^{t} H(t,\tau) x^n(\tau) d\tau = 0, \\ \displaystyle\int_{y(t)}^{t} K(t,\tau) x^n(\tau) d\tau = f(t), \end{cases} \qquad (2)$$

where $n \ge 2,\ 0 < t_0 \le t \le T,$ with given functions $H(t,\tau),\ K(t,\tau) \in C_{[t_0,\infty]\times[t_0,\infty]},\quad f(t) \in C_{[t_0,\infty]}$ and unknown functions $x(t) \in C_{[t_0,\infty]},\ y(t) \in C^1_{[t_0,\infty]}$ such that $y(t) < t$.

The aim of the work is to find the unknown functions $x(t)$ and $y(t)$ in (2). So, we introduce the operator notations

$$P(X) = (P_1(X), P_2(X)) = (0,0),\ X = (x(t), y(t)), \quad 0 < t_0 \le t \le T, \qquad (3)$$

then the system (2) on the interval $[t_0, T]$ can be reduced to the operator equation

$$\begin{cases} P_1(x(t), y(t)) = x(t) - \int_{y(t)}^{t} H(t,\tau)x^n(\tau)d\tau = 0, \\ P_2(x(t), y(t)) = f(t) - \int_{y(t)}^{t} K(t,\tau)x^n(\tau)d\tau = 0. \end{cases}$$

To solve (3), we use the modified Newton-Kantorovich method by writing the approximate equation in the form
$P'(X_0)(X - X_0) + P(X_0) = 0$, $X_0 = (x_0(t), y_0(t))$, (4)
where $X_0 = (x_0(t), y_0(t))$ is the initial point, and the derivative $P'(X_0)$ of the nonlinear operator $P(X)$ at the point X_0 is determined by the matrix

$$P'(X_0) = \begin{pmatrix} \left.\frac{\partial P_1}{\partial x}\right|_{(x_0,y_0)} & \left.\frac{\partial P_1}{\partial y}\right|_{(x_0,y_0)} \\ \left.\frac{\partial P_2}{\partial x}\right|_{(x_0,y_0)} & \left.\frac{\partial P_2}{\partial y}\right|_{(x_0,y_0)} \end{pmatrix}$$

Consequently, we have

$$\left.\begin{aligned} \left.\frac{\partial P_1}{\partial x}\right|_{(x_0,y_0)} \left(\Delta x(t)\right) + \left.\frac{\partial P_1}{\partial y}\right|_{(x_0,y_0)} \left(\Delta y(t)\right) = -P_1(x_0(t), y_0(t)), \\ \left.\frac{\partial P_2}{\partial x}\right|_{(x_0,y_0)} \left(\Delta x(t)\right) + \left.\frac{\partial P_2}{\partial y}\right|_{(x_0,y_0)} \left(\Delta y(t)\right) = -P_2(x_0(t), y_0(t)), \end{aligned}\right\}$$

where $\Delta x(t) = x_1(t) - x_0(t)$, $\Delta y(t) = y_1(t) - y_0(t)$ and $\left(x_0(t), y_0(t)\right)$ is the initial given point.

To find $x_1(t)$ and $y_1(t)$ we evaluate $P'(X_0)$ by the definition, i.e.

$$\left.\frac{\partial P_1}{\partial x}\right|_{(x_0,y_0)} = \lim_{s\to 0} \frac{P_1(x_0 + sx, y_0) - P_1(x_0, y_0)}{s}$$

$$= \lim_{s\to 0}\frac{1}{s}\begin{bmatrix} x_0(t) + sx(t) - \int_{y_0(t)}^{t} H(t,\tau)[x_0(\tau) + sx(\tau)]^n d\tau \\ -x_0(t) + \int_{y_0(t)}^{t} H(t,\tau)x_0^n(\tau)d\tau \end{bmatrix}$$

$$= x(t) - \int_{y_0(t)}^{t} H(t,\tau)nx_0^{n-1}(\tau)x(\tau)d\tau = P'_{1x}(x_0, y_0),$$

$$\left.\frac{\partial P_1}{\partial y}\right|_{(x_0,y_0)} = \lim_{s\to 0}\frac{P_1(x_0, y_0 + sy) - P_1(x_0, y_0)}{s}$$

$$= \lim_{s\to 0}\frac{\left[x_0(t) - \int_{y_0(t)+sy(t)}^{t} H(t,\tau)x_0^n(\tau)d\tau - x_0(t) + \int_{y_0(t)}^{t} H(t,\tau)x_0^n(\tau)d\tau\right]}{s}$$

$$= \lim_{s\to 0}\frac{\int_{y_0(t)}^{y_0(t)+sy(t)} H(t,\tau)x_0^n(\tau)d\tau}{s}$$

$$= H(t, y_0(t))x_0^n(y_0(t))y(t) = P'_{1y}(x_0, y_0).$$

Likewise,

$$\left.\frac{\partial P_2}{\partial x}\right|_{(x_0,y_0)} = -\int_{y_0(t)}^{t} K(t,\tau)nx_0^{n-1}(\tau)x(\tau)d\tau = P'_{2x}(x_0, y_0),$$

$$\left.\frac{\partial P_2}{\partial y}\right|_{(x_0,y_0)} = k(t, y_0(t))x_0^n(y_0(t))y(t) = P'_{2y}(x_0, y_0).$$

Therefore system (5) reduces to the system of linear Volterra integral equations

$$\left.\begin{aligned} &\Delta x(t) - \int_{y_0(t)}^{t} H(t,\tau)nx_0^{n-1}(\tau)\Delta x(\tau)d\tau + H(t, y_0(t))x_0^n(y_0(t))\Delta y(t) \\ &\quad = \int_{y_0(t)}^{t} H(t,\tau)x_0^n(\tau)d\tau - x_0(t), \\ &-\int_{y_0(t)}^{t} K(t,\tau)nx_0^{n-1}(\tau)\Delta x(\tau)d\tau + K(t, y_0(t))x_0^n(y_0(t))\Delta y(t) \\ &\quad = \int_{y_0(t)}^{t} K(t,\tau)x_0^n(\tau)d\tau - f(t). \end{aligned}\right\} \tag{6}$$

Solving (6) in terms of $\Delta x(t)$ and $\Delta y(t)$, we obtain $\left(x_1(t), y_1(t)\right)$. By continuing this process, we obtain a sequence of approximate solutions $\left(x_m(t), y_m(t)\right)$ from the following system:

$$\left.\begin{aligned} &\Delta x_m(t) - \int_{y_0(t)}^{t} H(t,\tau)nx_0^{n-1}(\tau)\Delta x_m(\tau)d\tau + H(t, y_0(t))x_0^n(y_0(t))\Delta y_m(t) \\ &\quad = \int_{y_{m-1}(t)}^{t} H(t,\tau)x_{m-1}^n(\tau)d\tau - x_{m-1}(t), \\ &-\int_{y_0(t)}^{t} K(t,\tau)nx_0^{n-1}(\tau)\Delta x_m(\tau)d\tau + K(t, y_0(t))x_0^n(y_0(t))\Delta y_m(t) \\ &\quad = \int_{y_{m-1}(t)}^{t} K(t,\tau)x_{m-1}^n(\tau)d\tau - f(t), \end{aligned}\right\} \tag{7}$$

where $\Delta x_m(t) = x_m(t) - x_{m-1}(t)$ and $\Delta y_m(t) = y_m(t) - y_{m-1}(t)$, $m = 2, 3, \ldots$.

Furthermore, by multiplying the first equation in (6) by $(-K(t, y_0(t))$ and the second equation by $H(t, y_0(t))$ and adding the two-equations together, we get

$$-\Delta x(t)+n\int_{y_0(t)}^{t}\left[H(t,\tau)-\frac{K(t,\tau)H(t,y_0(t))}{K(t,y_0(t))}\right]x_0^{n-1}(\tau)\Delta x(\tau)d\tau$$
$$=\int_{y_0(t)}^{t}\left[\frac{K(t,\tau)H(t,y_0(t))}{K(t,y_0(t))}-H(t,\tau)\right]x_0^{n}(\tau)d\tau+x_0(t)-f(t)\frac{H(t,y_0(t))}{K(t,y_0(t))}. \quad (8)$$

Let $G(t), K_1(t,\tau)$ and $F_0(t)$ be defined by

$$G(t)=\frac{H(t,y_0(t))}{K(t,y_0(t))},$$
$$K_1(t,\tau)=H(t,\tau)-K(t,\tau)G(t),$$
$$F_0(t)=\int_{y_0(t)}^{t}K_1(t,\tau)x_0^{n}(\tau)d\tau+f(t)G(t)-x_0(t).$$

Then Eq. (8) can be written in the form

$$\Delta x(t)-n\int_{y_0(t)}^{t}K_1(t,\tau)x_0^{n-1}(\tau)\Delta x(\tau)d\tau=F_0(t). \quad (9)$$

Eq. (9) is a linear Volterra integral equation of the second kind. It is known that the Eq. (9) has a unique solution (see [8, 9]). Solving Eq. (6) in terms of $\Delta y(t)$ yields

$$\Delta y(t)=\frac{\int_{y_0(t)}^{t}H(t,\tau)\left[x_0^{n}(\tau)+nx_0^{n-1}(\tau)\Delta x(\tau)\right]d\tau-(x_0(t)+\Delta x(t))}{H(t,y_0(t))x_0^{n}(y_0(t))} \quad (10)$$

Eq. (7) can be solved in the same way as solving (6), that is, performing similar operations as we have done for solving (6) we arrive at the equations in the form

$$\begin{cases}\Delta x_m(t)-n\int_{y_0(t)}^{t}K_1(t,\tau)x_0^{n-1}(\tau)\Delta x_m(\tau)d\tau=F_{m-1}(t),\\ \Delta y_m(t_i)=\frac{1}{H(t_i,y_0(t_i))x_0^{n}(y_0(t_i))}\left[n\int_{y_0(t_i)}^{t_i}H(t_i,\tau)x_0^{n-1}(\tau)\Delta x_m(\tau)d\tau-\Delta x_m(t_i)\right.\\ \qquad\left.+\int_{y_{m-1}(t_i)}^{t_i}H(t_i,\tau)x_{m-1}^{n}(\tau)d\tau-x_{m-1}(t_i)\right]\end{cases}$$

and therefore, we obtain a sequence of approximate solutions $X_m=(x_m(t),y_m(t))$.

Let us introduce the following classes of functions:

Let $C_{[t_0,\infty)}$ be a class of continuous functions $f(t)$ defined on the interval $[t_0,\infty)$.

Let $C_{[t_0,\infty)\times[t_0,\infty)}$ be a set of all continuous functions $\psi(t,u)$ defined on the interval $[t_0,\infty)\times[t_0,\infty)$.

Let $C^1_{t\,[t_0,\infty)}$ denote the class of continuous first derivative function defined on $[t_0,\infty)$, such that

$$C^1_{t\,[t_0,\infty)}=\left\{y(t)\in C^1_{[t_0,\infty)}:y(t)<t\right\}.$$

Let

$$\bar{C}=\left\{X=(x(t),y(t)):x(t),y(t)\in C_{[t_0,T]}\right\},$$

with the norms

$$\|\Delta X\|_{\bar{C}}=\max\left\{\|\Delta x\|_{C_{[t_0,T]}},\|\Delta y\|_{C_{[t_0,T]}}\right\},$$
$$\|X\|_{C^1}=\max\left\{\|x(t)\|,\|y(t)\|\right\},$$

where $\Delta x(t)=x(t)-x_0(t),\ \Delta y(t)=y(t)-y_0(t)$, and the norm is understood as Chebyshev norm.

The main result of the paper is

Theorem 1: Let $f(t)\in C_{[t_0,\infty]}$ and $H(t,\tau),K(t,\tau)\in C^1_{[t_0,\infty]\times[t_0,\infty]}$ such that $x(t)\in C^1_{[t_0,\infty)}$, $y(t)\in C^1_{t\,[t_0,\infty)}$, and let the ball $\Omega_0=\left\{X:\|X-X_0\|\le r\right\}$, if

i. $\|\Delta X\|\le\eta$, for all $x,y\in\Omega_0$,
ii. $K(t,\tau)$ and $H(t,\tau)$ are bounded and continuously differentiable functions on $[t_0,\infty)\times[t_0,\infty)$,
iii. $h=K\eta<\frac{1}{2}$, and $r<\frac{1+\sqrt{1-2h}}{h}\eta$, then system (2) has a unique solution X^*, and the sequence $\left\{X_m:X_m(t)=(x_m(t),y_m(t)),m\ge 0\right\}$ of successive approximations

$$\begin{cases}\Delta x_m(t)=n\int_{y_0(t)}^{t}K_1(t,\tau)x_0^{n-1}(\tau)\Delta x_m(\tau)d\tau+F_{m-1}(t),\\ \Delta y_m(t)=\frac{1}{H(t,y_0(t))x_0^{n}(y_0(t))}\left[\int_{y_{m-1}(t)}^{t}H(t,\tau)x_{m-1}^{n}(\tau)d\tau-\left(x_{m-1}(t)+\Delta x_m(t)\right)\right.\\ \qquad\left.+n\int_{y_0(t)}^{t}H(t,\tau)x_0^{n-1}(\tau)\Delta x_m(\tau)d\tau\right],\end{cases} \quad (11)$$

where $\Delta x_m(t)=x_m(t)-x_{m-1}(t)$ and $\Delta y_m(t)=y_m(t)-y_{m-1}(t)$, m=2, 3…, starting from $X_0=(x_0(t),y_0(t))$ converges to the unique solution X^*, and the rate of convergence is

$$\left\|X^*-X_m\right\|\le\frac{\eta}{h}\left(1-\sqrt{1-2h}\right)^{m+1},\qquad m\ge 0.$$

Proof: We define the operator

$$S(X)=X-\Gamma_0P(X).$$

Hence, the successive approximations of $X=S(X)$ is

$$X_{m+1}=S(X_m)\quad(m=0,1,...) \quad (12)$$

so that for the initial X_0 we have

$$S(X_0)=X_0-\Gamma_0P(X_0),$$

and therefore

$$\|\Gamma_0P(X_0)\|=\|S(X_0)-X_0\|=\|X_1-X_0\|=\|\Delta X\|\le\eta.$$

Now we prove $\|\Gamma_0P''(X)\|\le K$ for all $X\in\Omega_0$. The second derivative $P''(X_0)(X)$ of the nonlinear operator

$P(X)$ at the point X_0 is described by the 3-dimensional array $P''(X_0)X = (P_1, P_2)X$ in sense of bilinear operator.

$$P_1 = \begin{pmatrix} \left.\dfrac{\partial^2 P_1}{\partial x^2}\right|_{(x_0,y_0)} & \left.\dfrac{\partial^2 P_1}{\partial x\,\partial y}\right|_{(x_0,y_0)} \\ \left.\dfrac{\partial^2 P_1}{\partial y\,\partial x}\right|_{(x_0,y_0)} & \left.\dfrac{\partial^2 P_1}{\partial y^2}\right|_{(x_0,y_0)} \end{pmatrix},$$

$$P_2 = \begin{pmatrix} \left.\dfrac{\partial^2 P_2}{\partial x^2}\right|_{(x_0,y_0)} & \left.\dfrac{\partial^2 P_2}{\partial x\,\partial y}\right|_{(x_0,y_0)} \\ \left.\dfrac{\partial^2 P_2}{\partial y\,\partial x}\right|_{(x_0,y_0)} & \left.\dfrac{\partial^2 P_2}{\partial y^2}\right|_{(x_0,y_0)} \end{pmatrix}$$

All the second derivatives of $P_1''(X,Y)$ are computed by the following relations:

$$\left.\frac{\partial^2 P_1}{\partial x^2}\right|_{(x_0,y_0)} = \lim_{S\to 0}\frac{1}{s}\left[P_{1x}'(x_0+s\bar{x},y_0)(x,y) - P_{1x}'(x_0,y_0)(x,y)\right]$$

$$= \lim_{S\to 0}\frac{1}{s}\left[x(t) - \int_{y_0(t)}^{t} H(t,\tau)n[x_0(\tau)+s\bar{x}(\tau)]^{n-1}x(\tau)d\tau - x(t) + \int_{y_0(t)}^{t} H(t,\tau)nx_0^{n-1}(\tau)x(\tau)d\tau\right]$$

$$= -n(n-1)\int_{y_0(t)}^{t} H(t,\tau)x_0^{n-2}(\tau)\bar{x}(\tau)x(\tau)d\tau,$$

$$\left.\frac{\partial^2 P_1}{\partial x\partial y}\right|_{(x_0,y_0)} = \lim_{S\to 0}\frac{1}{s}\left[P_{1x}'(x_0,y_0+s\bar{y})(x,y) - P_{1x}'(x_0,y_0)(x,y)\right]$$

$$= \lim_{S\to 0}\frac{1}{s}\left[x(t) - \int_{y_0(t)+s\bar{y}(t)}^{t} H(t,\tau)nx_0^{n-1}(\tau)x(\tau)d\tau - x(t) + \int_{y_0(t)}^{t} H(t,\tau)nx_0^{n-1}(\tau)x(\tau)d\tau\right]$$

$$= nH(t,y_0(t))x_0^{n-1}(y_0(t))x(y_0(t))\bar{y}(t) \ ,$$

$$\left.\frac{\partial^2 P_1}{\partial y\partial x}\right|_{(x_0,y_0)} = \lim_{S\to 0}\frac{1}{s}\left[P_{1y}'(x_0+s\bar{x},y_0)(x,y) - P_{1y}'(x_0,y_0)(x,y)\right]$$

$$= \lim_{S\to 0}\frac{y(t)}{s}\left[H(t,y_0(t))[x_0(y_0(t))+s\bar{x}(y_0(t))]^n - H(t,y_0(t))x_0^n(y_0(t))\right]$$

$$= nH(t,y_0(t))x_0^{n-1}(y_0(t))\bar{x}(y_0(t))y(t) \ ,$$

$$\left.\frac{\partial^2 P_1}{\partial y^2}\right|_{(x_0,y_0)} = \lim_{S\to 0}\frac{1}{s}\left[P_{1y}'(x_0,y_0+s\bar{y})(x,y) - P_{1y}'(x_0,y_0)(x,y)\right]$$

$$= \left[H(t,y_0(t))x_0'^n(y_0(t)) + H'(t,y_0(t))x_0^n(y_0(t))\right]y(t) \ ,$$

In the same way we find the second derivatives of $P_2''(X,Y)$ by the definitions

$$\left.\frac{\partial^2 P_2}{\partial x^2}\right|_{(x_0,y_0)} = \lim_{S\to 0}\frac{1}{s}\left[P_{2x}'(x_0+s\bar{x},y_0)(x,y) - P_{2x}'(x_0,y_0)(x,y)\right]$$

$$= -n(n-1)\int_{y_0(t)}^{t} K(t,\tau)x_0^{n-2}(\tau)\bar{x}(\tau)x(\tau)d\tau;$$

$$\left.\frac{\partial^2 P_2}{\partial x\,\partial y}\right|_{(x_0,y_0)} = nK(t,y_0(t))x_0^{n-1}(y_0(t))x(y_0(t))\bar{y}(t) \ ;$$

$\dfrac{\partial^2 P_2}{\partial x\,\partial y}$ and $\dfrac{\partial^2 P_1}{\partial x\,\partial y}$ are valid since $\bar{y}(t)$ is bounded. $\dfrac{\partial^2 P_1}{\partial x^2}$, $\dfrac{\partial^2 P_2}{\partial x^2}$, $\dfrac{\partial^2 P_1}{\partial y\,\partial x}$ and $\dfrac{\partial^2 P_2}{\partial y\,\partial x}$ are valid since $\bar{x}(t)$ is bounded. Note that the norm of each component of the matrices $P_1(X_0,Y_0)$ and $P_2(X_0,Y_0)$ are bounded and all the conditions of Kantorovich theorem [8, page 532] are satisfied and hence:

(a) for $h = K\eta \le \frac{1}{2}$ and $r \ge r_0 = \dfrac{1-\sqrt{1-2h}}{h}\eta$. Eq. (4) has solution X^* and Newton-Kantorovich process Eq. (11) converges to X^*, and

(b) for $h = K\eta < \frac{1}{2}$ and $r < r_1 = \dfrac{1+\sqrt{1-2h}}{h}\eta$, X^* is a unique solution of Eq. (12) in Ω_0, and the rate of convergence is given by

$$\left\|X^* - X_m\right\|_{\bar{C}} \le \frac{\eta}{h}\left(1-\sqrt{1-2h}\right)^{m+1}, \qquad m = 0,1,2,...$$

This completes the proof of the Theorem 1.

$$\left.\frac{\partial^2 P_2}{\partial y\partial x}\right|_{(x_0,y_0)} = \lim_{S\to 0}\frac{1}{s}\left[P_{2y}'(x_0+s\bar{x},y_0)(x,y) - P_{2y}'(x_0,y_0)(x,y)\right]$$

$$= nK(t,y_0(t))x_0^{n-1}(y_0(t))\bar{x}(y_0(t))y(t) \ ;$$

$$\left.\frac{\partial^2 P_2}{\partial y^2}\right|_{(x_0,y_0)} = \lim_{S\to 0}\frac{1}{s}\left[P_{2y}'(x_0,y_0+s\bar{y})(x,y) - P_{2y}'(x_0,y_0)(x,y)\right]$$

$$= \left[K(t,y_0(t))x_0'^n(y_0(t)) + K'(t,y_0(t))x_0^n(y_0(t))\right]y(t) \ ;$$

3. RESULTS and DISCUSSION

Due to (6) and (9) it follows that the solution of (7) has the same procedure to be found depends on the having the solution of the equation

$$\Delta x_m(t) - n\int_{y_0(t)}^{t} K_1(t,\tau)x_0^{n-1}(\tau)\Delta x_m(\tau)d\tau = F_{m-1}(t), \quad (13)$$

where

$$K_1(t,\tau) = H(t,\tau) - K(t,\tau)G(t),$$

$$F_{m-1}(t) = x_{m-1}(t) - f(t)G(t) - \int_{y_{m-1}(t)}^{t} K_1(t,\tau)x_{m-1}^n(\tau)d\tau \cdot$$

Therefore, we have to solve the linear Volterra integral equation of the second kind. On the closed interval $[t_0, T]$, we introduce a grid of the points

$$t_i = t_0 + ih, \ h = \frac{T - t_0}{N}, \ i = 0,...,N \ .$$

Then Eq. (13) at the grid becomes

$$\Delta x_m(t_0) = F_{m-1}(t_0),$$

$$\Delta x_m(t_i) - \int_{y_0(t_i)}^{t_{v_i}} K_1(t_i,\tau) n x_0^{n-1}(\tau)\Delta x_m(\tau)d\tau - \sum_{j=v_i}^{i-1}\int_{t_j}^{t_{j+1}} K_1(t_i,\tau) n x_0^{n-1}(\tau)\Delta x_m(\tau)d\tau = F_{m-1}(t_i), \quad i=1,...,N. \tag{14}$$

Next let $v_i = [y_0(t_i)]$ and

$$t_{v_i} = \begin{cases} t_{v_i}, & t_0 < y_0(t_i) < t_{i-1}, \\ t_i, & t_{i-1} \le y_0(t_i) < t_i. \end{cases}$$

Then the system (14) can be represented in the form

$$\Delta x_m(t_i) - \int_{y_0(t_i)}^{t_{v_i}} K_1(t_i,\tau) n x_0^{n-1}(\tau)\Delta x_m(\tau)d\tau - \sum_{j=v_i}^{i-1}\int_{t_j}^{t_{j+1}} K_1(t_i,\tau) n x_0^{n-1}(\tau)\Delta x_m(\tau)d\tau = F_{m-1}(t_i), \quad i=1,...,N. \tag{15}$$

By computing the integrals in (15) by the trapezoidal formula we have two cases.

Case 1: If $v_i \ne i$, we obtain, for all $i = 1,2,...,N$

$$\Delta x_m(t_i) - \left(\frac{t_{v_i} - y_0(t_i)}{2}\right)\left(K_1(t_i,y_0(t_i)) n x_0^{n-1}(y_0(t_i))\Delta x_m(y_0(t_i)) + K_1(t_i,t_{v_i}) n x_0^{n-1}(t_{v_i})\Delta x_m(t_{v_i})\right)$$
$$- \sum_{j=v_i}^{i-2}\left(\frac{t_{j+1}-t_j}{2}\right)\left(K_1(t_i,t_j) n x_0^{n-1}(t_j)\Delta x_m(t_j) + K_1(t_i,t_{j+1}) n x_0^{n-1}(t_{j+1})\Delta x_m(t_{j+1})\right)$$
$$- \left(\frac{t_i - t_{i-1}}{2}\right)\left(K_1(t_i,t_{i-1}) n x_0^{n-1}(t_{i-1})\Delta x_m(t_{i-1}) + K_1(t_i,t_i) n x_0^{n-1}(t_i)\Delta x_m(t_i)\right) = F_{m-1}(t_i).$$

Solving this equation in terms of Δx_m yields

$$\Delta x_m(t_i) = \frac{F_{m-1}(t_i) + A + B + \left(\frac{t_i - t_{i-1}}{2}\right)\left(K_1(t_i,t_{i-1}) n x_0^{n-1}(t_{i-1})\Delta x_m(t_{i-1})\right)}{1 - \left(\frac{t_i - t_{i-1}}{2}\right)K_1(t_i,t_i) n x_0^{n-1}(t_i)}$$

Where

$$A = \left(\frac{t_{v_i} - y_0(t_i)}{2}\right)\Big[\, K_1(t_i,y_0(t_i)) n x_0^{n-1}(y_0(t_i))\left(\frac{(y_0(t_i) - t_{v_i-1})\Delta x_m(t_{v_i-1}) + (t_{v_i} - y_0(t_i))\Delta x_m(t_{v_i})}{t_{v_i} - t_{v_i-1}}\right)$$
$$+ K_1(t_i,t_{v_i}) n x_0^{n-1}(t_{v_i})\Delta x_m(t_{v_i})\Big],$$

$$B = \sum_{j=v_i}^{i-2}\left(\frac{t_{j+1}-t_j}{2}\right)\left(K_1(t_i,t_j) n x_0^{n-1}(t_j)\Delta x_m(t_j) + K_1(t_i,t_{j+1}) n x_0^{n-1}(t_{j+1})\Delta x_m(t_{j+1})\right)$$

Case 2: But if $v_i = i$, then

$$\Delta x_m(t_i) - \left(\frac{t_i - y_0(t_i)}{2}\right)\left[K_1(t_i,y_0(t_i)) n x_0^{n-1}(y_0(t_i))\Delta x_m(y_0(t_i)) + K_1(t_i,t_i) n x_0^{n-1}(t_i)\Delta x_m(t_i)\right] = F_{m-1}(t_i).$$

From the mean value theorem, we have

$$\Delta x_m(y_0(t_i))(t_i - t_{i-1}) = \int_{t_{i-1}}^{t_i} \Delta x_m(x)dx$$

By dividing the integral into two intervals $(t_{i-1}, y_0(t_i))$ and $(y_0(t_i), t_i)$, using the trapezoidal formula for each integral, we obtain

$$\Delta x_m(y_0(t_i)) = \frac{(y_0(t_i) - t_{i-1})\Delta x_m(t_{i-1}) + (t_i - y_0(t_i))\Delta x_m(t_i)}{t_i - t_{i-1}}$$

Therefore, (16) becomes

$$\Delta x_m(t_i) = \frac{1}{1 - \left(\frac{t_i - y_0(t_i)}{2}\right)K_1(t_i,t_i) n x_0^{n-1}(t_i)}\Bigg[\left(\frac{t_i - y_0(t_i)}{2}\right)K_1(t_i,y_0(t_i)) n x_0^{n-1}(y_0(t_i))\left(\frac{y_0(t_i) - t_{i-1}}{t_i - t_{i-1}}\right)\Delta x_m(t_{i-1})$$
$$- \left(\frac{t_i - y_0(t_i)}{2}\right)K_1(t_i,y_0(t_i)) n x_0^{n-1}(y_0(t_i))\left(\frac{t_i - y_0(t_i)}{t_i - t_{i-1}}\right) + F_{m-1}(t_i)\Bigg]$$

So, for $\Delta y_m(t)$ we again introduce a grid of points to (10) and therefore (10) becomes

$$\Delta y_m(t_i) = \frac{1}{H(t_i,y_0(t_i))x_0^n(y_0(t_i))}\Bigg[\int_{y_0(t_i)}^{t_i} H(t_i,\tau) n x_0^{n-1}(\tau)\Delta x_m(\tau)d\tau - \Delta x_m(t_i) + \int_{y_{m-1}(t_i)}^{t_i} H(t_i,\tau)x_{m-1}^n(\tau)d\tau - x_{m-1}(t_i)\Bigg] \tag{17}$$

And by applying Trapezoidal formula to Eq. (17), we obtain:

If $v_i \ne i$, then

$$\Delta y_m(t_i) = \frac{1}{H(t_i,y_0(t_i)){x^n}_0(y_0(t_i))}\Big[-\left(x_{m-1}(t_i) + \Delta x_m(t_i)\right)$$
$$+ \frac{\left(t_{v_i} - y_{m-1}(t_i)\right)}{2}\left(H(t_i,y_{m-1}(t_i)){x^n}_{m-1}(y_{m-1}(t_i)) + H(t_i,t_{v_i}){x^n}_{m-1}(t_{v_i})\right)$$
$$+ \frac{\left(t_{j+1} - t_j\right)}{2}\sum_{j=v_i}^{i-1}\left(H(t_i,t_j){x^n}_{m-1}(t_j) + H(t_i,t_{j+1}){x^n}_{m-1}(t_{j+1})\right)$$
$$+ \frac{\left(t_{v_i} - y_0(t_i)\right)}{2}\left(H(t_i,y_0(t_i)) n x_0^{n-1}(y_0(t_i))\Delta x_m(y_0(t_i)) + H(t_i,t_{v_i}) n x_0^{n-1}(t_{v_i})\Delta x_m(t_{v_i})\right)$$
$$+ \frac{\left(t_{j+1} - t_j\right)}{2}\sum_{j=v_i}^{i-1}\left(H(t_i,t_j) n x_0^{n-1}(t_j)\Delta x_m(t_j) + H(t_i,t_{j+1}) n x_0^{n-1}(t_{j+1})\Delta x_m(t_{j+1})\right)\Big]$$

and if $v_i = i$, then

$$\Delta y_m(t_i) = \frac{1}{H(t_i,y_0(t_i))x_0^n(y_0(t_i))}\Big[-\left(x_{m-1}(t_i) + \Delta x_m(t_i)\right)$$
$$+ \frac{\left(t_i - y_{m-1}(t_i)\right)}{2}\left(H(t_i,y_{m-1}(t_i))x_{m-1}^n(y_{m-1}(t_i)) + H(t_i,t_i)x_{m-1}^n(t_i)\right)$$
$$+ \frac{\left(t_i - y_0(t_i)\right)}{2}\left(H(t_i,y_0(t_i)) n x_0^{n-1}(y_0(t_i))\Delta x_m(y_0(t_i)) + H(t_i,t_i) n x_0^{n-1}(t_i)\Delta x_m(t_i)\right)\Big]$$

4. CONCLUSION

In this paper we have constructed the modified Newton-Kantorovich method (4) to solve the system (2) in terms of the unknown functions $x(t)$ and $y(t)$ by reducing (2) to linear VIE of the second kind. We have proven the existence and uniqueness of the solution of the system of NIEs. The Convergence theorem presented explains how the method converges and what the rate of convergence is.

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