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Complete Fuzzy n-normed linear space

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ABSTRACT

This paper introduces the notion of Cauchy sequence, convergent sequence and completeness in fuzzy n-normed linear space.

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| Fuzzy n-norm | Cauchy sequence | convergent sequence | completeness |

1. Introduction

Gahler [4] introduced the theory of n-norm on a linear space. For a systematic development of n-normed linear space one may refer to [5, 6, 8, 9]. In [5], Hendra Gunawan and Mashadi have also discussed the Cauchy sequence and convergent sequence in n-normed linear space. A detailed theory of fuzzy normed linear space can be found in [1, 2, 3, 7, 11]. In [10], we have extended n-normed linear space to fuzzy n-normed linear space.

Our object in this paper is to introduce the notion of Cauchy sequence and convergent sequence in fuzzy n-normed linear space and to study the completeness of the fuzzy n-normed linear space.

2. Preliminaries

Definition 2.1 [5]. Let $n \in N$ (natural numbers) and X be a real linear space of dimension $d \geq n$.(Here we allow d to be infinite). A real valued function $\|\bullet,...,\bullet\|$ on $X \times X \times \ldots \times X$ (n times)$=X^n$ satisfying the following four properties:

(1) $\| x_1, x_2,...,x_n \| = 0$ if any only if $x_1, x_2,...,x_n$ are linearly dependent

(2) $\| x_1, x_2,...,x_n \|$ is invariant under any permutation of $x_1, x_2,...,x_n$

(3) $\| x_1, x_2,...,cx_n \| = |c| \| x_1, x_2,...,x_n \|$, for any real c

(4) $\| x_1, x_2,...,x_{n-1},y + z \| \leq \| x_1, x_2,...,x_{n-1},y \| + \| x_1, x_2,...,x_{n-1}, z \|$

is called an n-norm on X and the pair (X, $\|\bullet,...,\bullet\|$) is called an n- normed linear space.

Definition 2.2 [5]. A sequence $\{x_n\}$ in an n-normed linear space (X, $\|\bullet,...,\bullet\|$) is said to converge to an $x \in X$ (in the n-norm) whenever $\lim_{n \to \infty} \| x_1, x_2,\ldots,x_{n-1}, x_n - x \| = 0$.

Definition 2.3 [5]. A sequence $\{x_n\}$ in an n-normed linear space (X, $\|\bullet,...,\bullet\|$) is called a Cauchy sequence if $\lim_{n, k \to \infty} \| x_1, x_2,\ldots, x_{n-1}, x_n - x_k \| = 0$.

Definition 2.4 [5]. An n-normed linear space is said to be complete if every Cauchy sequence in it is convergent.

Definition 2.5 [10]. Let X be a linear space over a real field F . A fuzzy subset N of $X^n \times R$ (R-set of real numbers) is called a fuzzy n-norm on X if and only if :

(N1) For all $t \in R$ with $t \leq 0$, $N (x_1, x_2,...,x_n, t) = 0$.

(N2) For all $t \in R$ with $t > 0$, $N (x_1, x_2,...,x_n ,t) = 1$ if and only if $x_1, x_2,...,x_n$ are linearly dependent.

(N3) $N(x_1,x_2,...,x_n,t)$ is invariant under any permutation of $x_1, x_2,...,x_n$.

(N4) For all $t\in R$ with $t > 0$,

$N(x_1,x_2,...,c\, x_n, t) = N(x_1, x_2,...,x_n, t /|c|)$ if $c \neq 0$, $c\in F$.

(N5) For all $s, t \in R$,

$N(x_1,x_2,..., x_n +x'_n , s + t) \geq \min \{ N(x_1,x_2,..., x_n , s), N(x_1,x_2,..., x'_n , t)\}$.

(N6) $N(x_1, x_2,...,x_n , t)$ is a non-decreasing function of $t\in R$ and $\lim_{t \to \infty} N(x_1,x_2,...,x_n,t) =1$.

Then (X , N) is called a fuzzy n-normed linear space or in short f-n-NLS.

Definition 2.6 [12]. A binary operation $*: [0,1] \times [0,1]\to [0,1]$ is called a continuous t-norm if * satisfies the following conditions:

(1) * is commutative and associative
(2) * is continuous
(3) $a*1=a$ for all $a \in[0,1]$
(4) $a* b \leq c* d$ whenever $a \leq c$ and $b \leq d$ and $a,b,c,d \in [0,1]$.

3. Complete fuzzy n-normed linear space

In this section we first redefine the notion of fuzzy n-normed linear space using t-norm.

Definition 3.1. Let X be a linear space over a real field F . A fuzzy subset N of $X^n \times [0, \infty)$ is called a fuzzy n-norm on X if and only if :

(N1$'$) $N(x_1, x_2,...,x_n, t) > 0$.

(N2$'$) $N(x_1, x_2,...,x_n, t) = 1$ if and only if $x_1, x_2,...,x_n$ are linearly dependent.

(N3$'$) $N(x_1,x_2,...,x_n,t)$ is invariant under any permutation of $x_1, x_2,...,x_n$.

(N4$'$) $N(x_1,x_2,...,c\, x_n, t) = N(x_1, x_2,...,x_n, t/|c|)$ if $c \neq 0$, $c \in F$(field).

(N5$'$) $N(x_1,x_2,..., x_n+x'_n , s + t) \geq N(x_1,x_2,..., x_n , s) * N(x_1,x_2,..., x'_n , t)$.

(N6$'$) $N(x_1, x_2,...,x_n , t)$ is left continuous and non-decreasing function such that $\lim_{t \to \infty} N(x_1,x_2,...,x_n,t) =1$.

Then (X , N) is called a fuzzy n-normed linear space or in short f-n-NLS.

To strengthen the above definition, we present the following example.

Example 3. 2. Let (X, $\|\bullet,...,\bullet\|$) be an n-normed linear space.

Define $a * b = \min\{a, b\}$ and $N(x_1, x_2,...,x_n, t) = t / (t + \| x_1, x_2,...,x_n\|)$.

Then (X, N) is a f-n-NLS.

Proof.

(N1$'$) Clearly $N(x_1, x_2,...,x_n, t) > 0$.

(N2$'$) $N(x_1, x_2,...,x_n, t) = 1$

$\Leftrightarrow$ $t / (t + \| x_1, x_2,...,x_n\|) =1$
$\Leftrightarrow$ $\| x_1, x_2,...,x_n\| = 0$
$\Leftrightarrow$ $x_1, x_2,...,x_n$ are linearly dependent.

(N3$'$) $N(x_1,x_2,...,x_n,t)$

$= t / (t + \| x_1, x_2,...,x_n\|)$
$= t / (t + \| x_1, x_2,...,x_n , x_{n-1}\|)$
$= N(x_1,x_2,...,x_n, x_{n-1} , t)$.

It follows similarly for the rest.

(N4′) $N(x_1,x_2,...,x_n,t / |c|)$

$= (t / |c|) / (t/ |c| + \| x_1, x_2,...,x_n \|)$
$= t / (t + |c| \| x_1, x_2,...,x_n \|)$
$= t / (t + \| x_1, x_2,...,cx_n \|)$
$= N(x_1,x_2,...,cx_n,t)$.

(N5′) Without loss of generality assume that $N(x_1,x_2,..., x'_n , t) \le N(x_1,x_2,..., x_n , s)$. Then

$t / (t +\| x_1,x_2,..., x'_n \|) \le s / (s +\| x_1,x_2,..., x_n \|)$

$\Rightarrow t (s +\| x_1,x_2,..., x_n \|) \le s (t +\| x_1,x_2,..., x'_n \|)$
$\Rightarrow t \| x_1,x_2,..., x_n \| \le s \| x_1,x_2,..., x'_n \|$
$\Rightarrow \| x_1,x_2,..., x_n \| \le (s / t) \| x_1,x_2,..., x'_n \|$.

Therefore,

$\| x_1,x_2,..., x_n \| + \| x_1,x_2,..., x'_n \|$

$\le (s / t) \| x_1,x_2,..., x'_n \| + \| x_1,x_2,..., x'_n \|$
$\le (s / t + 1) \| x_1,x_2,..., x'_n \|$
$= ((s +t) / t) \| x_1,x_2,..., x'_n \|$

But,

$\| x_1,x_2,..., x_n + x'_n \|$

$\le \| x_1,x_2,..., x_n \| + \| x_1,x_2,..., x'_n \|$
$\le ((s +t) / t) \| x_1,x_2,..., x'_n \|$.
$\Rightarrow (\| x_1,x_2,..., x_n + x'_n \|) / (s+t) \le (\| x_1,x_2,..., x'_n \|) / t$
$\Rightarrow 1 +(\| x_1,x_2,..., x_n + x'_n \|) / (s+t) \le 1 + (\| x_1,x_2,..., x'_n \|) / t$
$\Rightarrow ((s +t) + \| x_1,x_2,..., x_n + x'_n \|) / (s+t) \le (t+\| x_1,x_2,..., x'_n \|$
$\Rightarrow (s+t) / (\| x_1,x_2,..., x_n + x'_n \| +s+t) \ge t / (t+ \| x_1,x_2,..., x'_n \|)$
$\Rightarrow N(x_1,x_2,..., x_n +x'_n , s + t) \ge \min \{ N(x_1,x_2,..., x_n , s), N(x_1,x_2,..., x'_n , t)\}$.

(N6′) Clearly $N(x_1, x_2,...,x_n , t)$ is a left continuous function.

Suppose that $t_2 > t_1 > 0$ with $t_1 , t_2 \in [0, \infty)$ then,

$t_2 / (t_2 +\| x_1,x_2,..., x_n \|) - / t_1 (t_1 + \| x_1,x_2,..., x_n \|)$

$= \| x_1,x_2,..., x_n \| (t_2 - t_1) / ((t_2 +\| x_1,x_2,..., x_n \|) (t_1 + \| x_1,x_2,..., x_n \|))\ge 0$,

for all $(x_1,x_2,..., x_n)\in X^n$

$\Rightarrow t_2 / (t_2 +\| x_1,x_2,..., x_n \|) \ge / t_1 / (t_1 + \| x_1,x_2,..., x_n \|)$

$\Rightarrow N(x_1, x_2,\ldots,x_n, t_2) \geq N(x_1, x_2,\ldots,x_n, t_1)$.

Thus $N(x_1, x_2,\ldots,x_n, t)$ is a non-decreasing function of $t \in [0, \infty)$.

Also,

$$\lim_{t \to \infty} N(x_1,x_2,\ldots,x_n,t)$$

$$= \lim_{t \to \infty} t / (t + \| x_1,x_2,\ldots, x_n \|)$$

$$= \lim_{t \to \infty} t / t(1 + 1/t \| x_1,x_2,\ldots, x_n \|)$$

$$= 1.$$

Thus, (X, N) is a f-n-NLS.

Definition 3.3. A sequence $\{x_n\}$ in a f-n-NLS (X, N) is said to converge to x if given $r > 0$, $t > 0$, $0 < r < 1$, there exists an integer $n_0 \in N$ such that $N(x_1,x_2,\ldots, x_{n-1}, x_n-x, t) > 1-r$ for all $n \geq n_0$.

Theorem 3.4. In a f-n-NLS (X,N) a sequence $\{x_n\}$ converges to x if and only if $N(x_1,x_2,\ldots, x_{n-1}, x_n - x, t) \to 1$ as $n \to \infty$.

Proof.

Fix $t > 0$. Suppose $\{x_n\}$ converges to x.
Then for a given r, $0 < r < 1$, there exists an integer $n_0 \in N$ such that $N(x_1,x_2,\ldots, x_{n-1}, x_n - x, t) > 1- r$.
Thus $1-N(x_1,x_2,\ldots, x_{n-1}, x_n - x, t) < r$ and hence $N(x_1,x_2,\ldots, x_{n-1}, x_n - x, t) \to 1$ as $n \to \infty$.
Conversely, if for each $t > 0$, $N(x_1,x_2,\ldots, x_{n-1}, x_n - x, t) \to 1$ as $n \to \infty$, then for every r, $0 < r < 1$, there exists an integer n_0 such that $1- N(x_1,x_2,\ldots, x_{n-1}, x_n - x, t) < r$ for all $n \geq n_0$. Thus $N(x_1,x_2,\ldots, x_{n-1}, x_n - x, t) > 1-r$ for all $n \geq n_0$.
Hence $\{x_n\}$ converges to x in (X,N).

Definition 3.5. A sequence $\{x_n\}$ in a f-n-NLS (X,N) is said to be Cauchy sequence if given $\varepsilon > 0$ with $0 < \varepsilon < 1$, $t > 0$, there exists an integer $n_0 \in N$ such that $N(x_1,x_2,\ldots, x_{n-1}, x_n - x_k, t) > 1-\varepsilon$ for all $n, k \geq n_0$.

Theorem 3.6. In a f-n-NLS (X,N) every convergent sequence is a Cauchy sequence.

Proof. Let $\{x_n\}$ be a convergent sequence in (X, N). Suppose $\{x_n\}$ converges to x.
Let $t > 0$ and $\varepsilon \in (0,1)$. Choose $r \in (0,1)$ such that $(1-r) * (1-r) > 1-\varepsilon$.
Since $\{x_n\}$ converges to x, we have an integer n_0 such that $N(x_1,x_2,\ldots, x_{n-1}, x_n - x, t/2) > 1-r$.

Now, $N(x_1,x_2,\ldots, x_{n-1}, x_n - x_k, t)$
$= N(x_1,x_2,\ldots, x_{n-1}, x_n - x + x - x_k, t)$
$= N(x_1,x_2,\ldots, x_{n-1}, x_n - x, t/2) * N(x_1,x_2,\ldots, x_{n-1}, x - x_k, t/2)$

$\geq$ (1-r) * (1- r) for all n ,k $\geq n_0$
> 1-ε for all n ,k $\geq n_0$.

Therefore $\{x_n\}$ is a Cauchy sequence in (X, N).

Definition 3.7. A f-n-NLS is said to be complete if every Cauchy sequence in it is convergent.

The following example shows that there may exist Cauchy sequence in a f-n-NLS which is not convergent.

Example 3. 8. Let (X, ||•,...,•||) be an n-normed linear space and define a * b = min {a, b} for all a, b ∈ [0,1] and $N(x_1, x_2,...,x_n, t) = t / (t + \| x_1, x_2,...,x_n \|)$. Then (X, N) is shown to be a f-n-NLS.

Let $\{x_n\}$be a sequence in f-n-NLS, then

(a) $\{x_n\}$is a Cauchy sequence in (X, ||•,...,•||) if and only if $\{x_n\}$ is a Cauchy sequence in (X, N).
(b) $\{x_n\}$is a convergent sequence in (X, ||•,...,•||) if and only if $\{x_n\}$ is a convergent sequence in (X, N).

Proof.

(a) $\{x_n\}$is a Cauchy sequence in (X, ||•,...,•||)

$$\Leftrightarrow \lim_{n,k\to\infty} \| x_1,x_2,...,x_{n-1} ,x_n - x_k \| = 0.$$

$$\Leftrightarrow \lim_{n,k\to\infty} N (x_1,x_2,...,x_{n-1} ,x_n - x_k, t)$$

$$= \lim_{n,k\to\infty} t / (t +\| x_1,x_2,...,x_{n-1} ,x_n - x_k \|) = 1.$$

$$\Leftrightarrow N(x_1,x_2,..., x_{n-1}, x_n - x_k , t) \to 1 \quad \text{as} \quad n \to \infty.$$

$$\Leftrightarrow N(x_1,x_2,..., x_{n-1}, x_n - x_k , t)>1-r, \text{ for all } n ,k \geq n_0.$$

$\Leftrightarrow$ $\{x_n\}$ is a Cauchy sequence in (X, N).

(b) $\{x_n\}$is a convergent sequence in (X, ||•,...,•||)

$$\Leftrightarrow \lim_{n\to\infty} \| x_1,x_2,...,x_{n-1} ,x_n - x \| = 0.$$

$$\Leftrightarrow \lim_{n\to\infty} N (x_1,x_2,...,x_{n-1} ,x_n - x_k, t)$$

$$= \lim_{n\to\infty} t / (t +\| x_1,x_2,...,x_{n-1} ,x_n - x_k \|) = 1.$$

$\Leftrightarrow$ $N(x_1,x_2,..., x_{n-1}, x_n - x , t) \to 1$ as $n \to \infty$.

$\Leftrightarrow$ $N(x_1,x_2,..., x_{n-1}, x_n - x , t) > 1-r$, for all $n \geq n_0$.

$\Leftrightarrow$ $\{x_n\}$ is a convergent sequence in (X, N).

Thus if there exists an n-normed linear space (X, ||•,...,•||) which is not complete, then the fuzzy n-norm induced by such a crisp n-norm ||•,...,•|| on an incomplete n-normed linear space X is an incomplete fuzzy n-normed linear space.

Theorem 3.9. A f-n-NLS (X, N) in which every Cauchy sequence has a convergent subsequence is complete.

Proof.

Let $\{x_n\}$be a Cauchy sequence in (X, N) and $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ that converges to x. We prove that $\{x_n\}$ converges to x. Let $t > 0$ and $\varepsilon \in (0,1)$. Choose $r \in (0,1)$ such that $(1-r) * (1- r) > 1-\varepsilon$ **.** Since $\{x_n\}$ is a Cauchy sequence, there exists an integer $n_0 \in N$ such that $N(x_1,x_2,..., x_{n-1}, x_n - x_k , t / 2) > 1-r$ for all $n, k \geq n_0$. Since $\{x_{n_k}\}$ converges to x, there is a positive integer $i_k > n_0$ such that $N(x_1,x_2,..., x_{n-1}, x_{i_k} - x , t / 2) > 1-r$.

Now,

$$N(x_1,x_2,..., x_{n-1}, x_n - x , t / 2)$$

$$= N(x_1,x_2,..., x_{n-1}, x_n - x_{i_k} + x_{i_k} - x , t / 2 + t / 2)$$
$$\geq N(x_1,x_2,..., x_{n-1}, x_n - x_{i_k} , t / 2) * N(x_1,x_2,..., x_{n-1}, x_{i_k} - x , t / 2)$$
$$> (1-r) * (1- r)$$
$$> 1-\varepsilon.$$

Therefore $\{x_n\}$converges to x in (X, N) and hence it is complete.

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