Complete Fuzzy n-normed linear space

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ABSTRACT

This paper introduces the notion of Cauchy sequence, convergent sequence and completeness in fuzzy n-normed linear space.

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Fuzzy n-norm | Cauchy sequence | convergent sequence | completeness |

1. Introduction

Gahler [4] introduced the theory of n-norm on a linear space. For a systematic development of n-normed linear space one may refer to [5, 6, 8, 9]. In [5], Hendra Gunawan and Mashadi have also discussed the Cauchy sequence and convergent sequence in n-normed linear space. A detailed theory of fuzzy normed linear space can be found in [1, 2, 3, 7, 11]. In [10], we have extended n-normed linear space to fuzzy n-normed linear space.

Our object in this paper is to introduce the notion of Cauchy sequence and convergent sequence in fuzzy n-normed linear space and to study the completeness of the fuzzy n-normed linear space.

2. Preliminaries

Definition 2.1 [5]. Let n∈N (natural numbers) and X be a real linear space of dimension d≥n. (Here we allow d to be infinite). A real valued function ||, || on X × X ×…× X (n times)=X^n satisfying the following four properties:

(1) || x_1, x_2,...,x_n || = 0 if any only if x_1, x_2,...,x_n are linearly dependent

(2) || x_1, x_2,...,x_n || is invariant under any permutation of x_1, x_2,...,x_n
(3) \(|x_1, x_2, \ldots, cx_n| = |c| |x_1, x_2, \ldots, x_n|\), for any real \(c\)

(4) \(|x_1, x_2, \ldots, x_{n-1}, y + z| \leq |x_1, x_2, \ldots, x_{n-1}, y| + |x_1, x_2, \ldots, x_{n-1}, z|\)

is called an \(n\)-norm on \(X\) and the pair \((X, |\cdot, \ldots, \cdot|)\) is called an \(n\)-normed linear space.

**Definition 2.2 [5].** A sequence \(\{x_n\}\) in an \(n\)-normed linear space \((X, |\cdot, \ldots, \cdot|)\) is said to converge to an \(x \in X\) (in the \(n\)-norm) whenever \(\lim_{n \to \infty} |x_1, x_2, \ldots, x_{n-1}, x_n - x| = 0\).

**Definition 2.3 [5].** A sequence \(\{x_n\}\) in an \(n\)-normed linear space \((X, |\cdot, \ldots, \cdot|)\) is called a Cauchy sequence if
\[
\lim_{n, k \to \infty} |x_1, x_2, \ldots, x_{n-1}, x_n - x_k| = 0.
\]

**Definition 2.4 [5].** An \(n\)-normed linear space is said to be complete if every Cauchy sequence in it is convergent.

**Definition 2.5 [10].** Let \(X\) be a linear space over a real field \(F\). A fuzzy subset \(N\) of \(X^n \times \mathbb{R}\) (\(R\)-set of real numbers) is called a fuzzy \(n\)-norm on \(X\) if and only if:

(N1) For all \(t \in \mathbb{R}\) with \(t \leq 0\), \(N(x_1, x_2, \ldots, x_n, t) = 0\).

(N2) For all \(t \in \mathbb{R}\) with \(t > 0\), \(N(x_1, x_2, \ldots, x_n, t) = 1\) if and only if \(x_1, x_2, \ldots, x_n\) are linearly dependent.

(N3) \(N(x_1, x_2, \ldots, x_n, t)\) is invariant under any permutation of \(x_1, x_2, \ldots, x_n\).

(N4) For all \(t \in \mathbb{R}\) with \(t > 0\),
\[
N(x_1, x_2, \ldots, cx_n, t) = N(x_1, x_2, \ldots, x_n, t/|c|) \quad \text{if} \quad c \neq 0, \quad c \in F.
\]

(N5) For all \(s, t \in \mathbb{R}\),
\[
N(x_1, x_2, \ldots, x_n + x'_n, s + t) \geq \min \{N(x_1, x_2, \ldots, x_n, s), N(x_1, x_2, \ldots, x'_n, t)\}.
\]

(N6) \(N(x_1, x_2, \ldots, x_n, t)\) is a non-decreasing function of \(t \in \mathbb{R}\) and \(\lim_{t \to \infty} N(x_1, x_2, \ldots, x_n, t) = 1\).

Then \((X, N)\) is called a fuzzy \(n\)-normed linear space or in short f-n-NLS.

**Definition 2.6 [12].** A binary operation \(*: [0,1] \times [0,1] \to [0,1]\) is called a continuous \(t\)-norm if \(*\) satisfies the following conditions:

(1) \(*\) is commutative and associative

(2) \(*\) is continuous

(3) \(a^1 = a\) for all \(a \in [0,1]\)

(4) \(a^*b = c^*d\) whenever \(a \leq c\) and \(b \leq d\) and \(a, b, c, d \in [0,1]\).
3. Complete fuzzy n-normed linear space

In this section we first redefine the notion of fuzzy n-normed linear space using t-norm.

**Definition 3.1.** Let \( X \) be a linear space over a real field \( F \). A fuzzy subset \( N \) of \( X^n \times [0, \infty) \) is called a fuzzy n-norm on \( X \) if and only if:

(N1\(^{\prime}\)) \( N(x_1, x_2, \ldots, x_n, t) > 0 \).

(N2\(^{\prime}\)) \( N(x_1, x_2, \ldots, x_n, t) = 1 \) if and only if \( x_1, x_2, \ldots, x_n \) are linearly dependent.

(N3\(^{\prime}\)) \( N(x_1, x_2, \ldots, x_n, t) \) is invariant under any permutation of \( x_1, x_2, \ldots, x_n \).

(N4\(^{\prime}\)) \( N(x_1, x_2, \ldots, c x_n, t) = N(x_1, x_2, \ldots, x_n, 1/|c|) \) if \( c \neq 0, c \in F \) (field).

(N5\(^{\prime}\)) \( N(x_1, x_2, \ldots, x_n + x'_n, s + t) \geq N(x_1, x_2, \ldots, x'_n, s) \ast N(x_1, x_2, \ldots, x_n, t) \).

(N6\(^{\prime}\)) \( N(x_1, x_2, \ldots, x_n, t) \) is left continuous and non-decreasing function such that \( \lim_{t \to \infty} N(x_1, x_2, \ldots, x_n, t) = 1 \).

Then \( (X, N) \) is called a fuzzy n-normed linear space or in short f-n-NLS.

To strengthen the above definition, we present the following example.

**Example 3.2.** Let \( (X, \| \cdot \|, \ldots, \| \cdot \|) \) be an n-normed linear space.

Define \( a \ast b = \min \{a, b\} \) and \( N(x_1, x_2, \ldots, x_n, t) = t / (t + \|x_1, x_2, \ldots, x_n\|) \).

Then \( (X, N) \) is a f-n-NLS.

**Proof.**

(N1\(^{\prime}\)) Clearly \( N(x_1, x_2, \ldots, x_n, t) > 0 \).

(N2\(^{\prime}\)) \( N(x_1, x_2, \ldots, x_n, t) = 1 \)

\[ \iff t / (t + \|x_1, x_2, \ldots, x_n\|) = 1 \]

\[ \iff \|x_1, x_2, \ldots, x_n\| = 0 \]

\[ \iff x_1, x_2, \ldots, x_n \text{ are linearly dependent.} \]

(N3\(^{\prime}\)) \( N(x_1, x_2, \ldots, x_n, t) \)

\[ = t / (t + \|x_1, x_2, \ldots, x_n\|) \]

\[ = t / (t + \|x_1, x_2, \ldots, x_n, x_{n-1}\|) \]

\[ = N(x_1, x_2, \ldots, x_n, x_{n-1}, t) \).
It follows similarly for the rest.

\((N4')\) \(N(x_1,x_2,...,x_n,t / |c|)\)

\[= (t / |c|) / (t / |c| + \|x_1, x_2,...,x_n\|)\]

\[= t / (t + \|x_1, x_2,...,x_n\|)\]

\[= t / (t + \|x_1, x_2,...,cx_n\|)\]

\[= N(x_1,x_2,...,cx_nt).\]

\((N5')\) Without loss of generality assume that \(N(x_1,x_2,...,x_n', s) \leq N(x_1,x_2,...,x_n, s).\) Then

\[ t / (t + \|x_1,x_2,...,x_n'\|) \leq s / (s + \|x_1,x_2,...,x_n\|) \]

\[\Rightarrow t / (s + \|x_1,x_2,...,x_n\|) \leq s / (s + \|x_1,x_2,...,x_n'\|) \]

\[\Rightarrow \|x_1,x_2,...,x_n\| \leq (s / t) \|x_1,x_2,...,x_n'\|.\]

Therefore,

\[\|x_1,x_2,...,x_n\| + \|x_1,x_2,...,x_n'\| \]

\[\leq (s / t) \|x_1,x_2,...,x_n'\| + \|x_1,x_2,...,x_n'\| \]

\[= (s + t) / t \|x_1,x_2,...,x_n'\| \]

But,

\[\|x_1,x_2,...,x_n + x_n'\| \]

\[\leq \|x_1,x_2,...,x_n\| + \|x_1,x_2,...,x_n'\| \]

\[\leq (s + t) / t \|x_1,x_2,...,x_n'\|.\]

\[\Rightarrow (\|x_1,x_2,...,x_n + x_n'\|) / (s + t) \leq (\|x_1,x_2,...,x_n'\|) / t \]

\[\Rightarrow 1 + (\|x_1,x_2,...,x_n + x_n'\|) / (s + t) \leq 1 + (\|x_1,x_2,...,x_n'\|) / t \]

\[\Rightarrow (s + t) / (s + t) \|x_1,x_2,...,x_n + x_n'\| / (s + t) \leq (t + \|x_1,x_2,...,x_n'\|) \]

\[\Rightarrow (s + t) / (s + t) \|x_1,x_2,...,x_n + x_n'\| + t \geq t / (t + \|x_1,x_2,...,x_n'\|) \]

\[\Rightarrow N(x_1,x_2,...,x_n + x_n', s + t) \geq \min \{N(x_1,x_2,...,x_n, s), N(x_1,x_2,...,x_n', t)\}.\]

\((N6')\) Clearly \(N(x_1,x_2,...,x_n, t)\) is a left continuous function.

Suppose that \(t_2 > t_1 > 0\) with \(t_1, t_2 \in [0, \infty)\) then,

\[t_2 / (t_2 + \|x_1,x_2,...,x_n\|) \leq t_1 / (t_1 + \|x_1,x_2,...,x_n\|) \]

\[= \|x_1,x_2,...,x_n\| / (t_2 - t_1) \leq (t_2 + \|x_1,x_2,...,x_n\|) / (t_1 + \|x_1,x_2,...,x_n\|) \geq 0,\]

for all \((x_1,x_2,...,x_n) \in X^n\)

\[\Rightarrow t_2 / (t_2 + \|x_1,x_2,...,x_n\|) \geq t_1 / (t_1 + \|x_1,x_2,...,x_n\|) \]
\[ N(x_1, x_2, \ldots, x_n, t) \geq N(x_1, x_2, \ldots, x_n, t_1). \]

Thus \( N(x_1, x_2, \ldots, x_n, t) \) is a non-decreasing function of \( t \in [0, \infty) \).

Also,
\[
\lim_{t \to \infty} N(x_1, x_2, \ldots, x_n, t) = \lim_{t \to \infty} t / (t + \| x_1, x_2, \ldots, x_n \|) = \lim_{t \to \infty} t / t (1 + 1/t \| x_1, x_2, \ldots, x_n \|) = 1.
\]

Thus, \((X, N)\) is a f-n-NLS.

**Definition 3.3.** A sequence \( \{x_n\} \) in a f-n-NLS \((X, N)\) is said to converge to \( x \) if given \( r > 0, t > 0, 0 < r < 1 \), there exists an integer \( n_0 \in \mathbb{N} \) such that \( N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) > 1 - r \) for all \( n \geq n_0 \).

**Theorem 3.4.** In a f-n-NLS \((X, N)\) a sequence \( \{x_n\} \) converges to \( x \) if and only if \( N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) \to 1 \) as \( n \to \infty \).

**Proof.**
Fix \( t > 0 \). Suppose \( \{x_n\} \) converges to \( x \). Then for a given \( r, 0 < r < 1 \), there exists an integer \( n_0 \in \mathbb{N} \) such that \( N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) > 1 - r \).

Thus \( 1 - N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) < r \) and hence \( N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) \to 1 \) as \( n \to \infty \).

Conversely, if for each \( t > 0 \), \( N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) \to 1 \) as \( n \to \infty \), then for every \( r, 0 < r < 1 \), there exists an integer \( n_0 \) such that \( 1 - N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) < r \) for all \( n \geq n_0 \). Thus \( N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t) \to 1 - r \) for all \( n \geq n_0 \).

Hence \( \{x_n\} \) converges to \( x \) in \((X, N)\).

**Definition 3.5.** A sequence \( \{x_n\} \) in a f-n-NLS \((X, N)\) is said to be Cauchy sequence if given \( \varepsilon > 0 \) with \( 0 < \varepsilon < 1, t > 0 \), there exists an integer \( n_0 \in \mathbb{N} \) such that \( N(x_1, x_2, \ldots, x_{n-1}, x_n - x_k, t) > 1 - \varepsilon \) for all \( n, k \geq n_0 \).

**Theorem 3.6.** In a f-n-NLS \((X, N)\) every convergent sequence is a Cauchy sequence.

**Proof.** Let \( \{x_n\} \) be a convergent sequence in \((X, N)\). Suppose \( \{x_n\} \) converges to \( x \).

Let \( t > 0 \) and \( \varepsilon \in (0,1) \). Choose \( r \in (0,1) \) such that \( (1-r) * (1-r) > 1 - \varepsilon \).

Since \( \{x_n\} \) converges to \( x \), we have an integer \( n_0 \) such that \( N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t/2) > 1 - r \).

Now, \( N(x_1, x_2, \ldots, x_{n-1}, x_n - x_k, t) \)
\[ = N(x_1, x_2, \ldots, x_{n-1}, x_n - x + x_k - x_k, t) \]
\[ = N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t/2) * N(x_1, x_2, \ldots, x_{n-1}, x - x_k, t/2) \]

≥ (1-r) * (1-r) for all n,k ≥ n₀
> 1-ε for all n,k ≥ n₀.

Therefore \( \{xₙ\} \) is a Cauchy sequence in \((X, N)\).

**Definition 3.7.** A f-n-NLS is said to be complete if every Cauchy sequence in it is convergent.

The following example shows that there may exist Cauchy sequence in a f-n-NLS which is not convergent.

**Example 3.8.** Let \((X, ||\cdot,\cdot,\cdot||)\) be an n-normed linear space and define \(a * b = \min \{a, b\}\) for all \(a, b \in [0,1]\) and \(N(x₁, x₂,\ldots, xₙ, t) = t / (t + \|x₁, x₂,\ldots, xₙ\|)\). Then \((X, N)\) is shown to be a f-n-NLS.

Let \(\{xₙ\}\) be a sequence in f-n-NLS, then

(a) \(\{xₙ\}\) is a Cauchy sequence in \((X, ||\cdot,\cdot,\cdot||)\) if and only if \(\{xₙ\}\) is a Cauchy sequence in \((X, N)\).

(b) \(\{xₙ\}\) is a convergent sequence in \((X, ||\cdot,\cdot,\cdot||)\) if and only if \(\{xₙ\}\) is a convergent sequence in \((X, N)\).

**Proof.**

(a) \(\{xₙ\}\) is a Cauchy sequence in \((X, ||\cdot,\cdot,\cdot||)\)

\[\lim_{n, k \to \infty} \|x₁, x₂,\ldots, xₙ - xₖ\| = 0.\]

\[\lim_{n, k \to \infty} N(x₁, x₂,\ldots, xₙ - xₖ, t) = \lim_{n, k \to \infty} t / (t + \|x₁, x₂,\ldots, xₙ - xₖ\|) = 1.\]

\[\lim_{n, k \to \infty} N(x₁, x₂,\ldots, xₙ - xₖ, xₙ - xₖ) \to 1 \text{ as } n \to \infty.\]

\[\lim_{n, k \to \infty} N(x₁, x₂,\ldots, xₙ - xₖ, xₙ - xₖ) \to 1 - r, \text{ for all } n, k \geq n₀.\]

\(\{xₙ\}\) is a Cauchy sequence in \((X, N)\).

(b) \(\{xₙ\}\) is a convergent sequence in \((X, ||\cdot,\cdot,\cdot||)\)

\[\lim_{n \to \infty} \|x₁, x₂,\ldots, xₙ - x\| = 0.\]

\[\lim_{n \to \infty} N(x₁, x₂,\ldots, xₙ - x, t) = \lim_{n \to \infty} t / (t + \|x₁, x₂,\ldots, xₙ - x\|) = 1.\]
Thus if there exists an n-normed linear space \((X, \| \cdot, \cdot, \cdot \|)\) which is not complete, then the fuzzy n-norm induced by such a crisp n-norm \(\| \cdot, \cdot, \cdot \|\) on an incomplete n-normed linear space \(X\) is an incomplete fuzzy n-normed linear space.

**Theorem 3.9.** A f-n-NLS \((X, N)\) in which every Cauchy sequence has a convergent subsequence is complete.

**Proof.**

Let \(\{x_n\}\) be a Cauchy sequence in \((X, N)\) and \(\{x_{n_k}\}\) be a subsequence of \(\{x_n\}\) that converges to \(x\). We prove that \(\{x_{n_k}\}\) converges to \(x\). Let \(t > 0\) and \(\varepsilon \in (0,1)\). Choose \(r \in (0,1)\) such that \((1-r) * (1-r) > 1 - \varepsilon\). Since \(\{x_n\}\) is a Cauchy sequence, there exists an integer \(n_0 \in \mathbb{N}\) such that \(N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t / 2) > 1-r\) for all \(n, k \geq n_0\).

Since \(\{x_{n_k}\}\) converges to \(x\), there is a positive integer \(i_k > n_0\) such that \(N(x_1, x_2, \ldots, x_{n-1}, x_{i_k} - x, t / 2) > 1-r\).

Now,

\[
N(x_1, x_2, \ldots, x_{n-1}, x_n - x, t / 2)
\]

\[
= N(x_1, x_2, \ldots, x_{n-1}, x_n - x_{i_k} + x_{i_k} - x, t / 2 + t / 2)
\]

\[
\geq N(x_1, x_2, \ldots, x_{n-1}, x_n - x_{i_k}, t / 2) * N(x_1, x_2, \ldots, x_{n-1}, x_{i_k} - x, t / 2)
\]

\[
> (1-r) * (1-r)
\]

\[
> 1-\varepsilon.
\]

Therefore \(\{x_n\}\) converges to \(x\) in \((X, N)\) and hence it is complete.

4. **References**


