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RESEARCH ARTICLE

The Conjugation Degree on a Set of Metacyclic 3-Groups

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Abstract

Research on commutativity degree has been done by many authors since 1965. The commutativity degree is defined as the probability that two randomly selected elements in a group commute. In this research, an extension of the commutativity degree called the probability that an element of a group fixes a set Ωis explored. The group *G* in our scope is metacyclic 3-group and the set Ω consists of a pair of distinct commuting elements in the group *G* in which their orders satisfy a certain condition. Meanwhile, the group action used in this research is conjugation. The probability that an element of *G* fixes a set Ω, defined as the conjugation degree on a set is computed using the number of conjugacy classes. The result turns out to be 7/8 or 1, depending on the orbit and the order of Ω.

Keywords: Commutativity degree, conjugation action, conjugation degree, metacyclic groups

INTRODUCTION

Given two elements *x* and *y* in a group *G*. The probability that these two randomly selected elements commute is called the commutativity degree, which was first introduced by Erdos and Turan in their four series of research on the statistical group theory done in 1965 up to 1970. The definition of the commutativity degree of a group is given in the following.

Definition 1 (Erdos and Turan, 1965)
Let G be a finite group. The commutativity degree is the probability that two random elements (x, y) in G commute, defined as follows:

$$P(G)=\frac{|\{(x,y)\in G\times G \mid xy=yx\}|}{|G|^2}.$$

In their study (Erdos and Turan, 1965; Erdos and Turan, 1968; Erdos and Turan, 1967a; Erdos and Turan, 1967b), several problems on symmetric groups are also investigated. Later on, (Gustafson, 1973) discussed and proved that the probability of a random pair of elements can be computed by dividing the number of conjugacy classes with the size of the group. He also showed that $P(G)\le\frac{5}{8}$.

Since then, the concept of the probability have been generalized and extended by several authors (Pournaki and Sobhani, 2008; Alghamdi and F. G. Russo, 2012; Erfanian *et al*., 2007; Castelaz, 2010; Barzgar *et al*., 2016).

In this paper, one of the extensions, namely the probability that a group element fixes a set, denoted by $P_G(\Omega)$, is discussed. This probability has been introduced by (Omer *et al*., 2013). In this research, the probability that an element of a metacyclic 3-group fixes a set Ω is determined by using the conjugation action.

The set Ω is the set of a pair of distinct commuting elements in the group G in which their orders satisfy a certain condition, which can also be written as $\Omega=\{(x,y)\in G\times G:\mathrm{lcm}(|x|,|y|)=3, xy=yx, x\neq y\}\setminus\{(y,x)\}$.

The computation of the probability is started by first finding the elements of $G\times G$ that satisfy the condition of the set Ω. Next, the number of orbits under the same group action on the set Ω is determined. Later on, we will see that the probability that an element of G fixes the set Ω depends on the number of the orbits.

Next, some concepts related to this paper are given in the following two definitions.

Definition 2 (Rose, 1994) Metacyclic Group
A group G is called metacyclic if it has a cyclic normal subgroup H such that the quotient group G/H is also cyclic.

Definition 3 (Rotman, 2010) Group Action on a Set

Let G be any finite group and X be a set. A group G acts on X if there is a function $G \times X \to X$, such that

i. $(gh)x = g(hx), \forall g,h \in G, x \in X$.

ii. $1_G x = x, \forall x \in G$.

Next, the presentation of the metacyclic *p*-groups that is used in this paper is presented in the following theorem.

Theorem 1 (Basri, 2014) Let G be a non-abelian metacyclic *p*-group. Then G is one of the following:

Type 1: $G \cong \langle a,b \mid a^{p^{\alpha}} = b^{p^{\beta}} = 1, [b,a] = a^{p^{\alpha-\beta}} \rangle$, where p is an odd prime and $\alpha, \beta, \delta \in \ , \delta \le \alpha < 2\delta, \delta \le \beta, \delta \le \min\{\alpha - 1, \beta\}$.

Type 2: $G \cong \langle a,b \mid a^{p^{\alpha}} = 1, b^{p^{\beta}} = a^{p^{\alpha-\varepsilon}}, [b,a] = a^{p^{\alpha-\beta}} \rangle$, where p is an odd prime and α, β, δ, $\varepsilon \in$, $\delta + \varepsilon \le \alpha < 2\delta$, $\delta \le \beta$, $\alpha < \beta + \varepsilon$, $\delta \le \min\{\alpha - 1, \beta\}$.

In this paper, these two presentations of metacyclic *p*-groups, named as Type 1 and Type 2, *p*=3 are considered.

Since the number of the orbit is needed before the probability can be computed, the definition of orbit in a group action, specifically conjugation action, denoted as $O(x)$ is given in the following definition.

Definition 5 (Goodman, 2006) The Orbit

Let G act on a set X and $x \in X$. The orbit of x, denoted by $O(x)$, is the subset of X where $O(x) = \{gx : g \in G\} \subseteq X$. If a group G acts on X by conjugation, the orbit is given by $O(x) = \{gxg^{-1} : g \in G\} \subseteq X$. This orbit is also known as the conjugacy classes of x in G. The elements that belong to the same orbits have the same order.

Next, since the research focuses on conjugation action on a set, the probability that an element of a group has been defined as follows:

Definition 6 Conjugation Degree on a Set

Let G be a finite group and Ω a set of ordered pairs (x,y) in $G \times G$ such that $\text{lcm}(|x|,|y|) = p$, $xy = yx$ and $x \ne y$. If G acts on Ω by conjugation, the conjugation degree on the set Ω is given as

$$P_G(\Omega) = \frac{\left|\{(g,\omega) \in G \times \Omega : g\omega g^{-1} = \omega\}\right|}{|\Omega||G|}.$$

This means the action on the set Ω is the conjugation action.

Throughout this paper, the probability that an element of a group fixes a set will be named as the conjugation degree on a set.

In the next section, some previous studies on the commutativity degree and probability will be presented.

SOME PRELIMINARIES

This section provides some previous results that have been done by several authors on the extension of the commutativity degree and probability.

(Sherman, 1975) extended the concept of commutativity degree by introducing the probability that an automorphism of a finite group fixes an arbitrary element with the following definition.

Definition 7 (Sherman, 1975) Let G be a group. Let X be a non-empty set of G, where G is a group permutation of X. Then, the probability that an automorphism of a group fixes a random element from X is defined as follows:

$$P_G(X) = \frac{|\{(g,x) \mid gx = x \forall g \in G, x \in X\}|}{|X||G|}.$$

Next, (Omer *et al.*, 2013) extended the probability given by (Sherman, 1975) by introducing the probability that a group element fixes a set. Later, (El-Sanfaz, 2016) generalized the probability given by (Omer *et al.*, 2013) with the following definition.

Theorem 2 (Gustafson, 1973) Let G be a finite group and let $\Omega = \{(a,b) \in G \times G \,\|a| = |b| = 2, ab = ba\}$. Let G act on Ω. Then the probability that an element of a group G fixes the set Ω is given as

$$P_G(\Omega) = \frac{\left|\{(g,\omega) \in G \times \Omega : g\omega = \omega\}\right|}{|\Omega||G|}.$$

Next, the following theorem shows the simplified formula in computing the probability that an element of a group fixes a set under the group action of G onto the set Ω.

Theorem 3 (El-Sanfaz, 2016) Let G be a finite group and let $\Omega = \{(a,b) \in G \times G \,\|a| = |b| = 2, ab = ba\}$. Let G act on Ω.. Then the probability that an element of a group G fixes the set Ω is given by

$$P_G(\Omega) = \frac{K(\Omega)}{|\Omega|},$$

where $K(\Omega)$ is the number of orbits under the group action of G on Ω.

Throughout this paper, the conjugation degree of a metacyclic 3-group is computed, which will be presented in results and discussions.

RESEARCH METHDODOLOGY

In this section, the methodology of the research is presented and an extension of the commutativity degree of a group, defined as the conjugation degree on a set is explored. Note that in this research, the group G represents the metacyclic *p*-group, where p is the odd prime 3. Meanwhile, the set considered is Ω, a set of ordered pair of elements in G of the form (x,y), such that the $\text{lcm}(|x|,|y|) = p$, $xy = yx$, $x \ne y$ and if $(x,y) \in \Omega$, then $(y,x) \notin \Omega$.

In addition, the group action employed in this research is conjugation action. The computation of the conjugation degree on the set Ω is started by determining the order of each element in the group, whereby the elements that follow the restriction of the set Ω are considered. The presentation of metacyclic *p*-groups, where p is an odd prime given by (Basri, 2014) is referred.

The group's presentation is categorized as Type 1 and Type 2. Since the value p is 3, there are a total of two groups that are considered in this research, namely metacyclic 3-group of Type 1 and metacyclic 3-group Type 2. Thereafter, the elements in the group are gathered together by following the restrictions in the set Ω. Once the set Ω has been determined, the computation of the orbits in the set Ω is conducted. Next, the number of orbits in the set Ω is determined, by following the size of each orbit. Subsequently, the conjugation degree on the set Ω is computed by dividing the number of orbits with the size of the set Ω.

RESULTS AND DISCUSSIONS

In this section, the main result of this paper which is the conjugation degree on a set of metacyclic 3-group of Type 1 and metacyclic 3-group Type 2 are presented. Before that, six lemmas that will be used in the computation of the conjugation degree are given in the following.

Lemma 1 gives the form of all elements of order three in metacyclic 3-groups of Type 1.

Lemma 1 Let G be a metacyclic 3-group of Type 1, namely $G \cong \langle a,b : a^{3^\alpha} = b^{3^\beta} = 1, [b,a] = a^{3^{\alpha-\delta}} \rangle$ where $\alpha,\beta,\delta \in$, $\delta \le \alpha < 2\delta$, $\delta \le \beta, \delta \le \min\{\alpha-1,\beta\}$. Then, the elements of order three in G are $a^{3^{(\alpha-1)}i}, 1 \le i \le 2, b^{3^{(\beta-1)}j}, 1 \le j \le 2, a^{3^{(\alpha-1)}i}b^{3^{(\beta-1)}j}, 1 \le i \le 2, 1 \le j \le 2,$ where $\alpha \ge 3$ and $\beta \ge 2$.

Proof:

Suppose a and b are elements of G. Since $a^{3^\alpha} = 1$ and $b^{3^\beta} = 1$, thus $|a|$ divides 3^α and $|b|$ divides 3^β. Then, the elements of order three are described as follows: Two elements of G generated by the generator $a \in G$ are in the form of $a^{3^{(\alpha-1)}i}, 1 \le i \le 2,$ two elements of G generated by the generator $b \in G$ are in the form of $b^{3^{(\beta-1)}j}, 1 \le j \le 2,$ and four elements of G generated by the generator $ab \in G$ are in the form of $a^{3^{(\alpha-1)}i}b^{3^{(\beta-1)}j}, 1 \le i \le 2, 1 \le j \le 2.$ Therefore, there are eight elements of order three in G, where $\alpha \ge 3$ and $\beta \ge 2$.

Lemma 1 will be used to form the elements in the set Ω, since the size of the set Ω plays a crucial part in the computation of the conjugation degree on a set.

In Lemma 2, the size of the set Ω of metacyclic 3-groups of Type 1 is presented.

Lemma 2 Let G be a metacyclic 3-group of Type 1, namely $G \cong \{a,b : a^{3^\alpha} = b^{3^\beta} = 1, [b,a] = a^{3^{\alpha-\delta}}\}$ where $\alpha,\beta,\delta \in$, $\delta \le \alpha < 2\delta$, $\delta \le \beta, \delta \le \min\{\alpha-1,\beta\}$, and $\Omega = \{(x,y) \in G \times G : \mathrm{lcm}(|x|,|y|) = 3, xy = yx, x \ne y\} \setminus \{(x,y)\}$. Then, the order of $\Omega, |\Omega| = 36$.

Proof:

Suppose $x, y \in G$ and $\Omega = \{(x,y) \in G \times G : \mathrm{lcm}(|x|,|y|) = 3, xy = yx, x \ne y\} \setminus \{(x,y)\}$. By Lemma 1 there are eight elements of order three, namely $a^{3^{(\alpha-1)}i}, 1 \le i \le 2,$ $b^{3^{(\beta-1)}j}, 1 \le j \le 2,$ $a^{3^{(\alpha-1)}i}b^{3^{(\beta-1)}j}, 1 \le i \le 2, 1 \le j \le 2.$ Considering $(x,y) \in \Omega$ such that $|x| \in \{1,3\}$ and $|y| \in \{1,3\}$. Thus, there are three cases: Case (i) $|x| = 1, |y| = 3$, Case (ii) $|x| = 3, |y| = 1$, Case (iii) $|x| = |y| = 3$. Since if $(x,y) \in \Omega$ then $(y,x) \notin \Omega$, only one case is considered from (i) and (ii). Since the identity element commutes with all $g \in G$, there are a total of eight ordered pairs $(x,y) \in \Omega$, described as follows: $\{(1, a^{3^{(\alpha-1)}i}), 1 \le i \le 2, (1, b^{3^{(\beta-1)}j}), 1 \le j \le 2, (1, a^{3^{(\alpha-1)}i}b^{3^{(\beta-1)}j}), 1 \le i \le 2,$ $1 \le j \le 2\}$. Next, consider Case (iii). It is found that $8 \times 8 = 64$ pairs commute. However, since $x \ne y$ and $(y,x) \notin \Omega$, four pairs of (x,x) and (y,y), and 32 of (y,x) pairs are excluded. Thus, there are 28 ordered pairs left, namely $\{(a^{3^{(\alpha-1)}}, a^{3^{(\alpha-1)}2}), (a^{3^{(\alpha-1)}}, b^{3^{(\beta-1)}}),$ $(a^{3^{(\alpha-1)}}, b^{3^{(\beta-1)}2}), (a^{3^{(\alpha-1)}}, a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}}), (a^{3^{(\alpha-1)}}, a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}2}),$ $(a^{3^{(\alpha-1)}}, a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}}), (a^{3^{(\alpha-1)}}, a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}2}), (a^{3^{(\alpha-1)}2}, b^{3^{(\beta-1)}}),$ $(a^{3^{(\alpha-1)}2}, b^{3^{(\beta-1)}2}), (a^{3^{(\alpha-1)}2}, a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}}), (a^{3^{(\alpha-1)}2}, a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}2}),$ $(a^{3^{(\alpha-1)}2}, a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}}), (a^{3^{(\alpha-1)}2}, a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}2}), (b^{3^{(\beta-1)}}, b^{3^{(\beta-1)}2}),$ $(b^{3^{(\beta-1)}}, a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}}), (b^{3^{(\beta-1)}}, a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}2}), (b^{3^{(\beta-1)}}, a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}}),$ $(b^{3^{(\beta-1)}}, a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}2}), (b^{3^{(\beta-1)}2}, a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}}), (b^{3^{(\beta-1)}2}, a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}2}),$ $(b^{3^{(\beta-1)}2}, a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}}), (b^{3^{(\beta-1)}2}, a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}2}), (a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}}, a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}2}),$ $(a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}}, a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}}), (a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}}, a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}2}),$ $(a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}2}, a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}}), (a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}2}, a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}2}),$ $\{(a^{3^{(\alpha-1)}}, a^{3^{(\alpha-1)}2}),$ $(a^{3^{(\alpha-1)}}, b^{3^{(\beta-1)}}), (a^{3^{(\alpha-1)}}, b^{3^{(\beta-1)}2}), (a^{3^{(\alpha-1)}}, a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}}), (a^{3^{(\alpha-1)}}, a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}2}),$ $(a^{3^{(\alpha-1)}}, a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}}), (a^{3^{(\alpha-1)}}, a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}2}), (a^{3^{(\alpha-1)}2}, b^{3^{(\beta-1)}}),$ $(a^{3^{(\alpha-1)}2}, b^{3^{(\beta-1)}2}), (a^{3^{(\alpha-1)}2}, a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}}), (a^{3^{(\alpha-1)}2}, a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}2}),$ $(a^{3^{(\alpha-1)}2}, a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}}), (a^{3^{(\alpha-1)}2}, a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}2}), (b^{3^{(\beta-1)}}, b^{3^{(\beta-1)}2}),$ $(b^{3^{(\beta-1)}}, a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}}), (b^{3^{(\beta-1)}}, a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}2}), (b^{3^{(\beta-1)}}, a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}}),$ $(b^{3^{(\beta-1)}}, a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}2}), (b^{3^{(\beta-1)}2}, a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}}), (b^{3^{(\beta-1)}2}, a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}2}),$ $(b^{3^{(\beta-1)}2}, a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}}), (b^{3^{(\beta-1)}2}, a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}2}), (a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}}, a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}2}),$ $(a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}}, a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}}), (a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}}, a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}2}),$ $(a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}2}, a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}}), (a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}2}, a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}2}),$ $(a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}}, a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}2})\}$. Thus, a total of $8 + 28 = 36$ ordered pairs of Ω is found. From then it follows that $|\Omega| = 36$.

As mentioned in the introduction, the orbit of $\omega \in \Omega$ is the subset of the set Ω. If G acts on the set Ω by conjugation, the orbits is also called the conjugacy classes of ω, written as $\mathrm{cl}(\omega)$ or $\mathrm{cl}(x,y)$, where $\mathrm{cl}(x,y) = \{g(x,y)g^{-1} : g \in G\}$, $\omega = (x,y) \in \Omega$ and $K(\Omega)$ denotes the number of orbit.

In the following lemma, the number of orbit is computed for metacyclic 3-groups of Type 1. This result will later be used in proving Theorem 4.

Lemma 3 Let G be a metacyclic 3-group of Type 1, namely $G \cong \langle a,b : a^{3^\alpha} = b^{3^\beta} = 1, [b,a] = a^{3^{\alpha-\delta}} \rangle$ where $\alpha,\beta,\delta \in$, $\delta \le \alpha < 2\delta$, $\delta \le \beta, \delta \le \min\{\alpha-1,\beta\}$, and $\Omega = \{(x,y) \in G \times G : \mathrm{lcm}(|x|,|y|) = 3, . xy = yx, x \ne y\} \setminus \{(x,y)\}$. If G acts on Ω by conjugation, then the number of orbits in the set Ω,

$$K(\Omega) = \begin{cases} 14, & \text{when } \beta = \delta \\ 36, & \text{when } \beta > \delta \end{cases}.$$

Proof:

If G acts on Ω by conjugation, then there exists a function $\phi : G \times \Omega \to \Omega$ such that $\phi_g(\omega) = g(\omega)g^{-1}, \omega \in \Omega, g \in G$. By Lemma 2,

$|\Omega|=36$, and the orbit is written as $\mathrm{cl}(\omega)$ = cl $(x,y)=\{g(x,y)g^{-1}: g\in G\}$. Thus, the orbit of ω under conjugation action of G is divided into two cases: Case (i) $\beta=\delta$, and Case (ii) $\beta>\delta$.

Case (i). The orbit of Case (i) is described as follows: Consider $(x,y)\in G\times G$. There are three orbits of size one, $\{(1,a^{3^{(\alpha-1)}})\}$, $\{(1,a^{3^{(\alpha-1)}i}),i=2\}$, and $\{(a^{3^{(\alpha-1)}},a^{3^{(\alpha-1)}i},i=2\}$, where $\alpha\geq 3$, and there are 11 orbits of size three, $\{(1,a^{3^{(\alpha-1)}i}b^{3^{(\beta-1)}}),0\leq i\leq 2\}$, $\{(1,a^{3^{(\alpha-1)}i}b^{3^{(\beta-1)}j}),0\leq i\leq 2,j=2\}$, $\{(a^{3^{(\alpha-1)}i}b^{3^{(\beta-1)}},a^{3^{(\alpha-1)}}),0\leq i\leq 2,\}$, $\{(b^{3^{(\beta-1)}},b^{3^{(\beta-1)}j}),(a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}},a^{3^{(\alpha-1)}i}b^{3^{(\beta-1)}j}),(a^{3^{(\alpha-1)}i}b^{3^{(\beta-1)}},a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}j}),i=2, j=2\}$, $\{(b^{3^{(\beta-1)}},a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}})$, $(a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}},a^{3^{(\alpha-1)}i}b^{3^{(\beta-1)}})$, $(a^{3^{(\alpha-1)}i}b^{3^{(\beta-1)}}, b^{3^{(\beta-1)}})$, $i=2\}$, $\{(a^{3^{(\alpha-1)}i}b^{3^{(\beta-1)}},a^{3^{(\alpha-1)}j}),0\leq i\leq 2,j=2\}$, $\{(b^{3^{(\beta-1)}},a^{3^{(\alpha-1)}} b^{3^{(\beta-1)}j}),(a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}},b^{3^{(\beta-1)}j})$, $(a^{3^{(\alpha-1)}i}b^{3^{(\beta-1)}}a^{3(\alpha-1)i}b^{3^{(\beta-1)}j}),i=2,j=2\}$, $\{(b^{3^{(\beta-1)}}a^{3^{(\alpha-1)}i}b^{3^{(\beta-1)}j}),(a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}},a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}j})$ $(a^{3^{(\alpha-1)}i}b^{3^{(\beta-1)}},b^{3^{(\beta-1)}j})$, $i=2,j=2\}$, $\{(a^{3(\alpha-1)},a^{3^{(\alpha-1)}i}b^{3^{(\beta-1)}j}),0\leq i\leq 2,j=2\}$, $\{(a^{3(\alpha-1)i}b^{3^{(\beta-1)}j}, a^{3^{(\alpha-1)}j}),\ 0\leq i\leq 2,j=2\}$ and $\{(b^{3^{(\beta-1)}j},a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}j})$, $(a^{3^{(\alpha-1)}i}b^{3^{(\beta-1)}j},b^{3^{(\beta-1)}j})$, $(a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}j},a^{3(\alpha-1)i}b^{3^{(\beta-1)}j}),i=2,j=2\}$, where $\alpha\geq 3,\beta\geq 2$. From this it follows that $K(\Omega)=14$, when $\beta=\delta$.

Case (ii). The orbit of Case (ii) can be described as follows: Consider $(x,y)\in G\times G$, and all elements of order three, namely $a^{3^{(\alpha-1)}i},1\leq i\leq 2,b^{3^{(\beta-1)}j},1\leq j\leq 2,a^{3^{(\alpha-1)}i}b^{3^{(\beta-1)}j},1\leq i\leq 2,1\leq j\leq 2$, where $\alpha\geq 3$ and $\beta\geq 2$ are in $Z(G)$, thus commute with all $g\in G$. Thus, the orbits of ω are equal to themselves, i.e. $\mathrm{cl}(x,y)=\{(x,y)\},\forall(x,y)\in\Omega$. Therefore, there are 36 orbits of ω of size one. From this it follows that $K(\Omega)=36$, when $\beta>\delta$. The proof then follows.

Next, the following three lemmas give the elements of size three in metacyclic 3-groups of Type 2, as well as the size of the set Ω and the size of orbits of the same group.

In Lemma 4, the elements of order three for metacylic 3-groups of Type 2 are presented.

Lemma 4 Let G be a metacyclic 3-group of Type 2, namely $G\cong\langle a,b: a^{3^{\alpha}}=1,b^{3^{\beta}}=a^{p^{\alpha-\varepsilon}},[b,a]=a^{3^{\alpha-\delta}}\rangle$, where $\alpha,\ \beta,\ \delta,\ \varepsilon\in$, $\delta\leq\beta,\ \delta+\varepsilon\leq\alpha<2\delta,\alpha<\beta+\varepsilon,\delta\leq\min\{\alpha-1,\beta\}$. Then, the elements of order three in G is divided into two cases, with the given presentation. For Case (i), where $\alpha=\beta$, the elements are described as follows: $a^{3^{(\alpha-1)}i},1\leq i\leq 2$, $a^{3^{(\alpha-2)}i}b^{3^{(\beta-1)}},i=2,5,8$, $a^{3^{(\alpha-2)}i}b^{3^{(\beta-1)}j}$, $i=1,4,7,j=2$, where $\alpha\geq 3,\beta\geq 3$. For Case (ii) where $\alpha<\beta$, the elements are described as follows: $a^{3^{(\alpha-1)}i},1\leq i\leq 2$, $b^{3^{(\beta-1)}j},1\leq j\leq 2$, $a^{3^{(\alpha-1)}i}b^{3^{(\beta-1)}j},1\leq i\leq 2,1\leq j\leq 2$, where $\alpha\geq 3,\beta\geq 4$.

Proof:

Suppose a and b are elements of G. Since $a^{3^{\alpha}}=1$, thus $|a|$ divides 3^{α}. Then, the elements are described as follows:

Case (i) $\alpha=\beta$. Two elements of G generated by the generator $a\in G$ are in the form of $a^{3^{(\alpha-1)}i},1\leq i\leq 2$, three elements of G generated by the generator $ab\in G$ are in the form of $a^{3^{(\alpha-2)}i}b^{3^{(\beta-1)}},i=2,5,8$, and three elements of G generated by the generator $ab\in G$ are in the form of $a^{3^{(\alpha-2)}i}b^{3^{(\beta-1)}j},i=1,4,7,j=2$. Therefore, there are eight elements of order three, where $\alpha\geq 3,\beta\geq 3$.

Case (ii) $\alpha<\beta$. Two elements of G generated by the generator $a\in G$ are in the form of $a^{3^{(\alpha-1)}i},1\leq i\leq 2$, two elements of G generated by the generator $b\in G$ are in the form of $b^{3^{(\beta-1)}j},1\leq j\leq 2$, and four elements of G generated by the generator $ab\in G$ are in the form of $a^{3^{(\alpha-1)}i}b^{3^{(\beta-1)}j},1\leq i\leq 2,1\leq j\leq 2$. Therefore, there are eight elements of order three, where $\alpha\geq 3,\beta\geq 4$. The proof then follows.

Next, the size of the set of metacyclic 3-groups of Type 2 is found.

Lemma 5 Let G be a metacyclic 3-group of Type 2, namely $G\cong\langle a,b: a^{3^{\alpha}}=1,b^{3^{\beta}}=a^{p^{\alpha-\varepsilon}},[b,a]=a^{3^{\alpha-\delta}}\rangle$, where $\alpha,\beta,\delta,\varepsilon\in$, and $\Omega=\{(x,y)\in G\times G:\mathrm{lcm}(|x|,|y|)=3,xy=yx,x\neq y\}\setminus\{(x,y)\}$. Then, the order of Ω, $|\Omega|=36$.

Proof:

Suppose $x,y\in G$ and $\Omega=\{(x,y)\in G\times G:\mathrm{lcm}(|x|,|y|)=3$, $xy=yx,x\neq y$, if $(x,y)\in\Omega$ then $(y,x)\notin\Omega\}$. By Lemma 4, the elements of order three in G is divided into two cases.

Case (i). $\alpha=\beta$. Considering $(x,y)\in\Omega$ such that $|x|\in\{1,3\}$ and $|y|\in\{1,3\}$, there are 3 cases: (ia) $|x|=1,|y|=3$, (ib) $|x|=3,|y|=1$, (ic) $|x|=|y|=3$. Since $(y,x)\notin\Omega$, only one case is considered from (ia) and (ib). Next, since identity element commutes with all $g\in G$, there are eight ordered pairs $(x,y)\in\Omega$ described as follows: $\{(1,a^{3^{(\alpha-1)}i}),\ 1\leq i\leq 2,\ (1,a^{3^{(\alpha-2)}i}b^{3^{(\beta-1)}}),\ i=2,5,8,\ i=1,4,7,j=2\}$ $(1,a^{3^{(\alpha-2)}i}b^{3^{(\beta-1)}j}),\ i=1,4,7,j=2\}$ where $\alpha\geq 3,\beta\geq 3$. Next, consider case (ic), it is found that $8\times 8=64$ pairs commute. However, since $x\neq y$ and $(y,x)\notin\Omega$, 4 pairs of (x,x) and (y,y), and 32 pairs of (y,x) were excluded. Thus, there are 28 pairs left, described as follows: $\{(a^{3^{(\alpha-1)}},a^{3^{(\alpha-1)}2})$, $(a^{3^{(\alpha-1)}},b^{3^{(\beta-1)}})$, $(a^{3^{(\alpha-1)}},b^{3^{(\beta-1)}2})$, $(a^{3^{(\alpha-1)}},a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}})$, $(a^{3^{(\alpha-1)}},a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}2})$, $(a^{3^{(\alpha-1)}},a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}})$, , $(a^{3^{(\alpha-1)}},a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}2})$, $(a^{3^{(\alpha-1)}2},b^{3^{(\beta-1)}})$, $(a^{3^{(\alpha-1)}2},b^{3^{(\beta-1)}2})$, $(a^{3^{(\alpha-1)}2},a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}})$, $(a^{3^{(\alpha-1)}2},a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}2})$, $(a^{3^{(\alpha-1)}2},a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}})$, $(a^{3^{(\alpha-1)}2},a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}2})$, $(b^{3^{(\beta-1)}},b^{3^{(\beta-1)}2})$, $(b^{3^{(\beta-1)}},a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}})$, $(b^{3^{(\beta-1)}},a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}2})$, $(b^{3^{(\beta-1)}2},a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}})$, $(b^{3^{(\beta-1)}2},a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}2})$, $(b^{3^{(\beta-1)}2},a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}})$, $(b^{3^{(\beta-1)}2},a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}2})$, $(a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}},a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}2})$, $(a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}},a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}})$, $(a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}},a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}2})$, $(a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}},a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}})$, $(a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}},a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}2})$, $(a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}},a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}2})\}$, where $\alpha\geq 3,\beta\geq 3$. Therefore, a total of

$8+28=36$ ordered pairs of Ω is found. From then, it follows that $|\Omega|=36$.

Case (ii). $\alpha<\beta$. Considering $(x,y)\in\Omega$, such that $|x|\in\{1,3\}$ and $|y|\in\{1,3\}$, there are 3 cases: (iia) $|x|=1, |y|=3$, (iib) $|x|=3, |y|=1$, (iic) $|x|=|y|=3$. Since $(x,y)=(y,x)$, only one case is considered from (iia) and (iib). Since identity element commutes with all $g\in G$, there are eight ordered pairs $(x,y)\in\Omega$ described as follows: $\{(1,a^{3^{(\alpha-1)}i}),1\le i\le 2,(1,b^{3^{(\beta-1)}j}),1\le j\le 2,(1,a^{3^{(\alpha-1)}i}b^{3^{(\beta-1)}j}),1\le i\le 2,1\le j\le 2\}$. Next, consider case (iic), it is found that $8\times 8=64$ pairs commute. However, since $x\neq y$ and if $(x,y)\in\Omega$ then $(y,x)\notin\Omega$, 4 pairs of (x,x) and (y,y), and 32 pairs of (y,x) were excluded. Thus, there are 28 pairs left, described as follows:

$\{(a^{3^{(\alpha-1)}},a^{3^{(\alpha-1)}2}), (a^{3^{(\alpha-1)}},b^{3^{(\beta-1)}}), (a^{3^{(\alpha-1)}},b^{3^{(\beta-1)}2}), (a^{3^{(\alpha-1)}},a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}}),$
$(a^{3^{(\alpha-1)}},a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}2}), (a^{3^{(\alpha-1)}},a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}}), (a^{3^{(\alpha-1)}},a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}2}),$
$(a^{3^{(\alpha-1)}2},b^{3^{(\beta-1)}}), (a^{3^{(\alpha-1)}2},b^{3^{(\beta-1)}2}), (a^{3^{(\alpha-1)}2},a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}}),(a^{3^{(\alpha-1)}2},a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}2}),$
$(a^{3^{(\alpha-1)}2}, a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}}), (a^{3^{(\alpha-1)}2},a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}2}), (b^{3^{(\beta-1)}},b^{3^{(\beta-1)}2}),$
$(b^{3^{(\beta-1)}},a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}}), (b^{3^{(\beta-1)}},a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}2}), (b^{3^{(\beta-1)}},a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}}),$
$(b^{3^{(\beta-1)}},a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}2}), (b^{3^{(\beta-1)}2},a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}}), (b^{3^{(\beta-1)}2},a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}2}),$
$(b^{3^{(\beta-1)}2},a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}}), (b^{3^{(\beta-1)}2},a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}2}), (a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}},a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}2}),$
$(a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}},a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}}), (a^{3^{(\alpha-1)}}b^{3^{(\beta-1)}},a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}2}),$
$(a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}},a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}}), (a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}},a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}2}),$
$(a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}},a^{3^{(\alpha-1)}2}b^{3^{(\beta-1)}2})\}$, where $\alpha\ge 3,\beta\ge 4$. Therefore, a total of 8+28=36 ordered pairs of Ω is found. From then, it follows that $|\Omega|=36$.

Next, the number of orbits, $K(\Omega)$ of metacyclic 3-groups of Type 2 is given in the following lemma. This result will later be used in proving Theorem 5.

Lemma 6 Let G be a metacyclic 3-group of Type 2, namely $G\cong\langle a,b:a^{3^\alpha}=1,b^{3^\beta}=a^{p^{\alpha-\varepsilon}},[b,a]=a^{3^{\alpha-\delta}}\rangle$, and $\Omega=\{(x,y)\in G\times G:\text{lcm}(|x|,|y|)=3,xy=yx,x\neq y\}\setminus\{(x,y)\}$. If G acts on Ω by conjugation, then the number of orbits in the set Ω,

$$K(\Omega)=\begin{cases}14, & \text{when } \alpha=\beta,\\ 36, & \text{when } \alpha<\beta.\end{cases}$$

Proof:

If G acts on Ω by conjugation, then there exists a function $\phi:G\times\Omega\to\Omega$ such that $\phi_g(\omega)=g\omega g^{-1},\omega\in\Omega,g\in G$. By Lemma 5, $|\Omega|=36$, and the orbit is written by $\text{cl}(\omega)$ = cl $(x,y)=\{g(x,y)g^{-1}:g\in G\}$. Thus, the orbits under conjugation action of G on Ω is divided into two cases: Case (i), $\alpha=\beta>\delta>\varepsilon$ and Case (ii), $\alpha<\beta>\delta>\varepsilon$.

Case (i). The orbits of Case (i) is described as follows: Consider $(x,y)\in G\times G$. There are three orbits of size one, in the form of $\{(1,a^{3^{(\alpha-1)}i})\}$, $i=1$, $\{(1,a^{3^{(\alpha-1)}i})\}$, $i=2$, and $\{(a^{3^{(\alpha-1)}i},a^{3^{(\alpha-1)}j})\}$, $i=1,j=2$ where $\alpha\ge 3,\beta\ge 3$, and there are 11 orbits of size three, in the form of $\{(1,a^{3^{(\alpha-2)}i}b^{3^{(\beta-1)}j}),\}$ $i=2,5,8,j=1$, $i=1,4,7,j=2\}$, $\{(a^{3^{(\alpha-2)}i},a^{3^{(\alpha-2)}j}b^{3^{(\beta-1)}k}),i=1,j=2,5,8,k=1\}$, $\{(a^{3^{(\alpha-2)}i},a^{3^{(\alpha-2)}j}b^{3^{(\beta-1)}k}),$ $i=1,j=1,4,7,k=2\}$, $\{(a^{3^{(\alpha-2)}i},a^{3^{(\alpha-2)}j}b^{3^{(\beta-1)}k}),$ $i=2,j=2,5,8,k=1\}$, $\{(a^{3^{(\alpha-2)}i},a^{3^{(\alpha-2)}j}b^{3^{(\beta-1)}k}),i=2,j=1,4,7,k=2\}$
$\{(a^{3^{(\alpha-2)}i}b^{3^{(\beta-1)}j},a^{3^{(\alpha-2)}k}b^{3^{(\beta-1)}m}),$ $i=2,j=1,k=5,m=1,$
$(a^{3^{(\alpha-2)}i}b^{3^{(\beta-1)}j},a^{3^{(\alpha-2)}k}b^{3^{(\beta-1)}m}),$ $i=5,j=1,k=8,m=1,$
$(a^{3^{(\alpha-2)}i}b^{3^{(\beta-1)}j},a^{3^{(\alpha-2)}k}b^{3^{(\beta-1)}m}),$ $i=8,j=1,k=2,$ $m=1\}$,
$\{(a^{3^{(\alpha-2)}i}b^{3^{(\beta-1)}j},a^{3^{(\alpha-2)}k}b^{3^{(\beta-1)}m}),$ $i=2,j=1,k=8,m=2,$
$(a^{3^{(\alpha-2)}i}b^{3^{(\beta-1)}j},a^{3^{(\alpha-2)}k}b^{3^{(\beta-1)}m}),i=5,j=1,k=7,m=2,(a^{3^{(\alpha-2)}i}(a^{3^{(\alpha-2)}i}$
$b^{3^{(\beta-1)}j},a^{3^{(\alpha-2)}k}b^{3^{(\beta-1)}m}),i=5,j=1,k=7,m=2,(a^{3^{(\alpha-2)}i}b^{3^{(\beta-1)}j},a^{3^{(\alpha-2)}k}$
$b^{3^{(\beta-1)}m}),i=2,j=1,k=5,m=2\}$, $\{(a^{3^{(\alpha-2)}i}b^{3^{(\beta-1)}j},a^{3^{(\alpha-2)}k}b^{3^{(\beta-1)}m}),$
$i=2,j=1,k=4,m=2,$ $(a^{3^{(\alpha-2)}i}b^{3^{(\beta-1)}j},a^{3^{(\alpha-2)}k}b^{3^{(\beta-1)}m}),$ $i=5,j=1,$
$k=1,m=2,(a^{3^{(\alpha-2)}i}b^{3^{(\beta-1)}j},a^{3^{(\alpha-2)}k}b^{3^{(\beta-1)}m}),i=8,j=1,k=7,m=2\}$,
$\{(a^{3^{(\alpha-2)}i}b^{3^{(\beta-1)}j},a^{3^{(\alpha-2)}k}b^{3^{(\beta-1)}m}),i=2,j=1,k=7,m=2,$ $(a^{3^{(\alpha-2)}i}b^{3^{(\beta-1)}j},$
$a^{3^{(\alpha-2)}k}b^{3^{(\beta-1)}m}),i=5,j=1,k=4,m=2,(a^{3^{(\alpha-2)}i}b^{3^{(\beta-1)}j},a^{3^{(\alpha-2)}k}b^{3^{(\beta-1)}m}),$
$i=8,j=1,k=1,m=2\}$, $\{(a^{3^{(\alpha-2)}i}b^{3^{(\beta-1)}j},a^{3^{(\alpha-2)}k}b^{3^{(\beta-1)}m}),$ $i=1,$ $j=1,$
$k=4,$ $m=2,$ $(a^{3^{(\alpha-2)}i}b^{3^{(\beta-1)}j},a^{3^{(\alpha-2)}k}b^{3^{(\beta-1)}m}),$ $i=7,$ $j=2,$ $k=1,$ $m=2,$
$(a^{3^{(\alpha-2)}i}b^{3^{(\beta-1)}j},a^{3^{(\alpha-2)}k}b^{3^{(\beta-1)}m}),i=4,j=2,k=7,m=2\}$, where $\alpha\ge 3,$ $\beta\ge 3$.

Case (ii). The orbit of Case (ii) can be described as follows: Consider $(x,y)\in G\times G$, and all elements of order three, namely $a^{3^{(\alpha-1)}i},1\le i\le 2,$ $b^{3^{(\beta-1)}j},1\le j\le 2$, $a^{3^{(\alpha-1)}i}b^{3^{(\beta-1)}j},1\le i\le 2,1\le j\le 2$, where $\alpha\ge 3,\beta\ge 4$, are in $Z(G)$, thus commute with all $g\in G$. Thus, the orbits of ω are equal to themselves, i.e. $\text{cl}(x,y)=\{(x,y)\},\forall(x,y)\in\Omega$. Therefore, there are 36 orbits of ω of size one. From this it follows that $K(\Omega)=36$ when $\alpha<\beta>\delta>\varepsilon$.

THE CONJUGATION DEGREE OF METACYCLIC 3-GROUPS

Based on the six lemmas given, the conjugation degree on a set of metacyclic 3-groups of Type 1 and Type 2 are computed, presented in the following two theorems.

The following theorem gives the conjugation degree on a set for metacyclic 3-group of Type 1.

Theorem 4 Let G be a metacyclic 3-group of Type 1, namely $G\cong\langle a,b:a^{3^\alpha}=b^{3^\beta}=1,[b,a]=a^{3^{\alpha-\delta}}\rangle$ where $\alpha,\beta,\delta\in\ ,\delta\le\alpha<2\delta,\delta\le\beta,\delta\le\min\{\alpha-1,\beta\}$, and $\Omega=\{(x,y)\in G\times G:\text{lcm}(|x|,|y|)=3,xy=yx,x\neq y\}\setminus\{(x,y)\}$. If G acts on Ω by conjugation, then the conjugation degree on the set Ω is given by

$$P_G(\Omega)=\begin{cases}\dfrac{7}{18}, & \text{when } \beta=\delta,\\ 1, & \text{when } \beta>\delta.\end{cases}$$

Proof.

By Lemma 2, $|\Omega| = 36$. Next, using Lemma 3, the number of orbits, $K(\Omega) = 14$, when $\beta = \delta$, and $K(\Omega) = 36$, when $\beta > \delta$. Thus, using Theorem 3, the conjugation degree on the set Ω, $P_G(\Omega) = \frac{K(\Omega)}{|\Omega|} = \frac{14}{36} = \frac{7}{18}$, when $\beta = \delta$, and $P_G(\Omega) = \frac{36}{36} = 1$, when $\beta > \delta$.

Theorem 5 gives the conjugation degree on a set for metacyclic 3-group of Type 2.

Theorem 5 Let G be a metacyclic 3-group of Type 2, namely $G \cong \langle a,b : a^{3^\alpha} = 1, b^{3^\beta} = a^{p^{\alpha-\varepsilon}}, [b,a] = a^{3^{\alpha-\delta}} \rangle$ and $\Omega = \{(x,y) \in G \times G : \text{lcm}(|x|,|y|) = 3, xy = yx, x \neq y\} \setminus \{(x,y)\}$. If G acts on Ω by conjugation, then the conjugation degree on the set Ω is given by $P_G(\Omega) = \begin{cases} \frac{7}{18}, & \text{when } \alpha = \beta, \\ 1, & \text{when } \alpha < \beta. \end{cases}$

Proof:

By Lemma 5, $|\Omega| = 36$. Next, using Lemma 6, the number of orbits, $K(\Omega) = 14$, when $\alpha = \beta$, and $K(\Omega) = 36$, when $\alpha < \beta > \delta > \varepsilon$. Thus, using Theorem 3, the conjugation degree on the set Ω, $P_G(\Omega) = \frac{K(\Omega)}{|\Omega|} = \frac{14}{36} = \frac{7}{18}$ when $\alpha = \beta$ and $P_G(\Omega) = 1$ when $\alpha = \beta$.

CONCLUSION

In this paper, the conjugation degree on a set is found for metacyclic 3-groups of Type 1 and Type 2. The result shows that the conjugation degree on a set for both groups are either $\frac{7}{8}$ or 1, depending on the parameters α, β, γ,Ú. It was also found that the conjugation degree on a set for both groups depend on the size of the orbits and the size of the set Ω.

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