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RESEARCH ARTICLE

Plane wave solution of extended discrete nonlinear Schrödinger equation

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Abstract

In this paper, we considered the extended discrete nonlinear Schrödinger equation (EDNLSE) which includes the nearest neighbour nonlinear interaction in addition to the on-site cubic and quintic nonlinearities. The objective of this study is to investigate the modulational instability of plane matter-wave solution in dipolar Bose-Einstein Condensates (BEC) in a periodic optical lattice and to compare the analytical results with numerical. Analytically, the problem is solved by using perturbed solution of the plane wave where the instability of the gain can be obtained. The conditions of the stability of the plane wave had been analysed and confirmed numerically, by applications of Runge-Kutta method. Three specific cases were studied where only cubic-quintic nonlinearity$(q = 0)$ is considered, only quintic-dipolar $(\alpha = 0)$ is considered and lastly non-zero for all terms. The numerical results are aligned with the analytical results.

Keywords: Discrete Nonlinear Schrödinger Equation, Periodic Potential, Modulational Instability, Numerical Method

INTRODUCTION

The study of nonlinear physics is a very interesting and expanding topic in modern applied mathematics due to its vast applications in physics (Yang J., 2010). One particular equation, the Discrete Nonlinear Schrödinger Equation (DNLSE) has attracted researchers' interest because of its importance in explaining discrete model lattice in many different fields of physics such as nonlinear optics (Mandelik D. *et al.,* 2004), matter waves, and photonic crystal (Brazhnyi V. A. and Konotop V. V., 2004). The DNLSE is relevant to many areas due to its wide range of phenomenon that the equations hold within, including Peierls-Nabarro potentials, discrete localised modes, and modulational instability (Kevrikidis P. G. & Carretero-González, R., 2009). The main focus of this paper is to study the particular extended DNLSE(EDNLSE) where the on-site cubic and quintic nonlinearities in addition to nearest neighbour nonlinear interaction are considered (Umarov B. A. *et al*., 2017). In its normalised, dimensionless units, it can be written as

$$i u_{n,t} + \kappa(u_{n+1} + u_{n-1}) + g|u_n|^2 u_n + q(|u_{n+1}|^2 + |u_{n-1}|^2)u_n + \alpha|u_n|^4 u_n = 0 \qquad (1)$$

where i is the imaginary unit, $u_n(t)$ is the complex dependent variable at site n, n is discrete variable taking integer values, t is evolution parameter (time), and t in subscript means derivative with respect t, parameters κ is for tunnelling, g for cubic nonlinearity, q for nonlocal interaction, and α for quintic nonlinearity. The equation is reduced to the standard DNLSE when $q = \alpha = 0$. The equation becomes cubic-quintic DNLSE when $q = 0$ and its solution have been studied by Abdullaev *et al.* (2007). Some of the previous literature such as in Rojas-Rojas *et al.* (2011) have studied the dynamics of dipolar Bose-Einstein Condensates in deep optical lattice which is the case for $\alpha = 0$. The EDNLSE that is being considered in this study includes combined effect of quintic and nonlocal nonlinearity, in addition to the cubic nonlinearity. The study of this model can be applied to cases where all two-body, three-body, and dipolar Bose-Einstein Condensates are to be considered such as in the works of Abdullaev *et al.* (2001), Trombettoni and Smerzi (2001), and Fattori *et al.* (2008). It is also convenient to study light beam propagation in the system of nonlinearity coupled optical waveguides when both cubic and quintic nonlinearities have to be considered as in the study conducted by Öster and Johansson (2005). The EDNLSE(1) has two conserved quantities, which are the norm

$$N = \sum_n |u_n|^2 \qquad (2)$$

and the Hamiltonian,

$$H = \sum_n \left[\kappa(u_n^* u_{n+1} + u_n u_{n+1}^*) + \frac{g}{2}|u_n|^4 + \frac{\alpha}{3}|u_n|^6 + q|u_{n+1}|^2|u_n|^2\right] \quad (3)$$

These quantities can be helpful to maintain a precision of numerical calculations. Modulational instabilities in different models have been studied thoroughly by others such as Kivshar and Peyrard (1992); Abdullaev et al. (2007) and Baizakov *et al*. (2009). The results on the modulational instability of nonlinear plane wave solution of EDNLSE are studied analytically and numerically. The numerical results for cases where cubic-dipolar DNLSE and cubic-quintic dipolar DNLSE is to be studied and compared to the analytical results.

Modulational Instability (MI) of nonlinear plane wave

To study the plane wave solution of equation (1), and by referring to the model introduced by B. A. Umarov et al. (2017), first insert the plane wave ansatz $u_n = u_0 e^{i(kn+\omega t)}$ into equation (1) to obtain the nonlinear dispersion relation

$$\omega = 2\kappa \cos(k) + (g + 2q)u_0^2 + \alpha u_0^4 \quad (4)$$

To study the stability of the plane wave solution, perturbation is introduced to the ansatz in the form of $u_n = [u_0 + \delta u_n(t)]e^{i(kn+\omega t)}$ and substitute into equation (1). By linearisation and set

$$\delta u_n(t) = u_1 e^{i(Qn+\Omega t)} + u_2^* e^{-i(Qn+\Omega^* t)} \quad (5)$$

then solving for the linear system, the equation of instability gain can be written as

$$\Omega = \frac{1}{2}\left[d \pm \sqrt{d^2 - 4b^2 + 4(\omega - a^+)(\omega - a^-)}\right] \quad (6)$$

where

$$d = a^+ - a^-,$$
$$a^\pm = 2(g+q)u_0^2 + 2\kappa\cos(Q \pm k) + 2qu_0^2\cos(Q) + 3\alpha u_0^4,$$
$$b = gu_0^2 + 2qu_0^2\cos(Q) + 2\alpha u_0^4.$$

The instability gain is defined as $G = \mathrm{Im}(\Omega)$ and nonlinear plane wave experiences modulational instability when $G \neq 0$. One can easily analyse equation (6) by setting some sets of parameters into the equation. By setting $q = 0$ for equation (1), the MI growth of cubic-quintic nonlinearity (Abdullaev *et al*., 2007) can be obtained such that

$$\Omega_{cq} = 4[\cos k\,(\cos Q - 1)(2\alpha u_0^2 + gu_0^2 + \cos k\,(\cos Q - 1)]^{1/2} \quad (7)$$

Then, by setting $\alpha = 0$, the MI growth of cubic-dipolar nonlinearity (Rojas-Rojas et al,2011) can be written as

$$\Omega_{cd} = 4[\cos k\,(\cos Q - 1)(2qu_0^2\cos Q + gu_0^2 + \cos k\,(\cos Q - 1)]^{1/2} \quad (8)$$

Finally, let's consider where all the parameters in equation (1) are present ($g \neq 0, q \neq 0, \alpha \neq 0$), then the MI growth is written as in equation (6) and substitute for $k = 0$ where we consider a uniform plane wave solution, can be simplified to

$$\Omega_{cdq} = 4[(cosQ - 1)(2\alpha u_0{}^4 + 2qu_0{}^2 cosQ + gu_0{}^2 + cosQ - 1)]^{1/2} \quad (9)$$

The onset of MI in equation (7), equation (8), and equation (9) is determined by the expression under the square root is negative. In this paper, we will focus on analysing the conditions for stability of equation (9). We only consider the value of Q to be positive since the growth rate of equation (9) is even function of perturbation wave number Q. From there we can see that for uniform plane wave ($k = 0$) is unstable whenever $2\alpha u_0{}^4 + 2qu_0{}^2 + gu_0{}^2 > 0$ which leads to following cases:

I. If $q > 0, g, \alpha < 0$, then MI will occur when $q > (|2\alpha u_0^2| + |g|)/2$.
II. If $q < 0, g, \alpha > 0$, then MI will occur when $|q| < (2\alpha u_0^2 + g)/2$.

Domains of stability corresponding to MI gain is shown in Fig. 1 (refer to the colour bar).

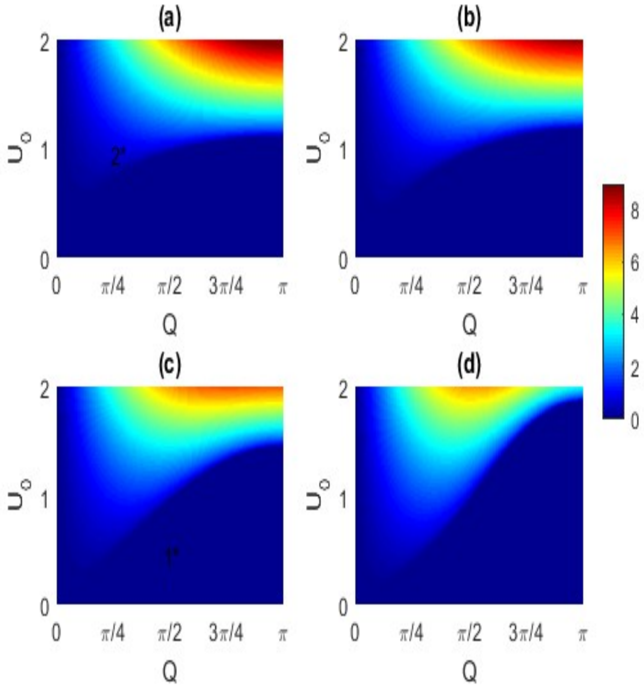

Fig. 1 Gain for $u_0$vs Q. G for nonlinear plane-wave solution for $g = 0.5, \alpha = 0.3, \kappa = 1$ and (a) $q = -0.2$,(b) $q = -0.0.5$, (c) $q = 0.4$ and (d) $q = 1.0$

RESULTS AND DISCUSSION

Numerical simulation of modulational instability

The study of the linear stability analysis is done so that we can identify particular regions in the parameter space where MI may occur. This also helps in estimating the perturbations growth rate by referring to equation (6). But the further evolutions of the wave field could not be predicted by linear stability theory. That is why numerical simulations is done so that the evolution of the wave field under MI and pattern formation of the system can be studied. The numerical simulation of equation (1) has been done using the fourth-order Runge-Kutta method with a small-time step so that the conserved quantities which is the Hamiltonian and Norm can be monitored to the accuracy of 10^{-4}. First the initial plane wave is set as $\overline{u_0} = u_0 + \delta u_0$. The perturbation is $\delta u_0 = 0.0001\cos Q$ where $Q = 2P\pi/n$ with P is an integer and n is the total number of lattice sites. In this simulation the value is set as $n = 100, \kappa = 1, k = 0$. The total time is set to $t = 300$. The stability and instability regions, as presumed by the analytical results of equation (9) were checked for different points which labelled as 1^* and 2^* in Fig 1. The parameters are set corresponds to the area of point 1^*, the MI gain is negligible and the plane wave remain unchanged as shown in Fig 2. But as we can see if the parameters are set corresponds to point 2^*, the plane wave is unstable and form some localised modes approximately after $t = 230$ as shown in Fig. 3.

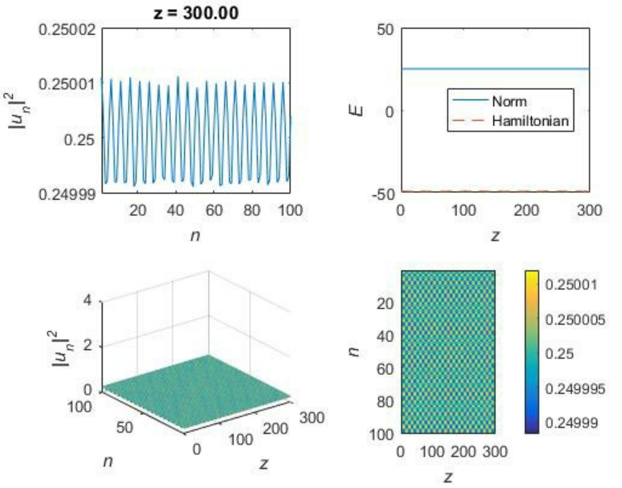

Fig. 2 The Evolution of weakly perturbed plane wave solution. The parameters are set $g = 0.5, \alpha = 0.3$ and $q = -0.4$. The MI gain is negligible and the wave propagation is stable.

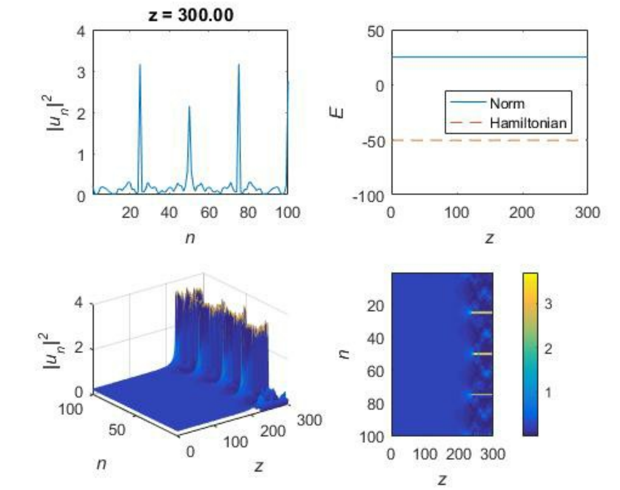

Fig. 3 The Evolution of weakly perturbed plane wave solution. The parameters are set $g = 0.5, \alpha = 0.3$ and $q = -0.2$. The MI gain is nonzero and the wave propagation is unstable.

CONCLUSION

In conclusion, the modulational instabilities in cubic-quintic with dipolar DNLSE have been investigated. The study of the stability regions in the system has been done analytically and numerically proven. The main objective is to study the effect of dipolar on the system. The condition of the stability of the plane wave solution has been analysed and checked numerically for $q > 0, g, \alpha < 0$ and $q < 0, g, \alpha > 0$. There are cases where the modulated waves are stable and unstable. The instability of the plane wave leads to the forming of localised modes. In the future, there are more analytical and numerical can be done with different parameters and also different types of solutions yet to be study for EDNLSE.

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REFERENCES

Abdullaev, F. K., Baizakov, B. B., Darmanyan, S. A., Konotop, V. V., Salerno, M. (2001). Nonlinear excitations in arrays of Bose-Einstein condensates. *Physical Review A, 64*(4).

Abdullaev, F., Bouketir, A., Messikh, A., Umarov, B. (2007). Modulational instability and discrete breathers in the discrete cubic–quintic nonlinear Schrödinger equation. *Physica D: Nonlinear Phenomena, 232*(1), 54-61.

Baizakov, B. B., Bouketir, A., Messikh, A., Umarov, B. A. (2009). Modulational instability in two-component discrete media with cubic-quintic nonlinearity. *Physical Review E, 79*(4).

Brazhnyi, V. A., Konotop, V. V. (2004). Theory of nonlinear matter waves in optical lattices. *Modern Physics Letters B, 18*(14), 627-651.

Fattori, M., D'Errico, C., Roati, G., Zaccanti, M., Jona-Lasinio, M., Modugno, M., Modugno, G. (2008). Atom interferometry with a weakly interacting bose-einstein condensate. *Physical Review Letters,100* (8).

Kevrekidis, P. G., Carretero-González, R. (2009). *The discrete nonlinear Schrödinger equation: Mathematical analysis, numerical computations and physical perspectives*. Berlin: Springer.

Kivshar, Y. S., Peyrard, M. (1992). Modulational Instabilities in Discrete Lattices. *Physical Review A, 46*(6), 3198-3205.

Mandelik, D., Morandotti, R., Aitchison, J. S., Silberberg, Y. (2004). Gap solitons in waveguide arrays. *Physical Review Letters, 92*(9).

Rojas-Rojas, S., Vicencio, R. A., Molina, M. I., Abdullaev, F. K. (2011). Nonlinear localized modes in dipolar bose-einstein condensates in optical lattices. *Physical Review A, 84*(3).

Trombettoni, A., Smerzi, A. (2001). Discrete solitons and breathers with dilute bose-einstein condensates. *Physical Review Letters, 86*(11), 2353-2356.

Umarov, B. A., Ismail, N. H., Hadi, M. S., Hassan, T. H. (2017). Localized solutions of extended discrete nonlinear schrödinger equation. *Journal of Physics: Conference Series, 890*, 012072.

Yang, J. (2010). Nonlinear waves in integrable and nonintegrable systems. *Philadelphia (Pa.): Society for Industrial and Applied Mathematics.*

Öster, M., Johansson, M. (2005). Phase twisted modes and current reversals in a lattice model of waveguide arrays with nonlinear coupling. *Physical Review E, 71*(2).