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RESEARCH ARTICLE

Rock-Paper-Scissors Lattice Model

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Abstract

In this work, we introduce Rock-Paper-Scissors lattice model on Cayley tree of second order generated by Rock-Paper-Scissors game. In this strategic 2-player game, the rule is simple: rock beats scissors, scissors beat paper, and paper beats rock. A payoff matrix A of this game is a skew-symmetric. It is known that quadratic stochastic operator generated by this matrix is non-ergodic transformation. The Hamiltonian of Rock-Paper-Scissors Lattice Model is defined by this skew-symmetric payoff matrix A. In this paper, we discuss a connection between three fields of research: evolutionary games, quadratic stochastic operators, and lattice models of statistical physics. We prove that a phase diagram of the Rock-Paper-Scissors model consists of translation-invariant and periodic Gibbs measure with period 3.

Keywords: Rock-Paper-Scissors game, quadratic stochastic operator, lattice model, Cayley tree

INTRODUCTION

Ising and Potts Models

According to Kindermann and Snell (1980), the Ising model is formulated by considering infinite graph $\Gamma=(V,\Lambda)$, where V is a countable set of vertices or sites and Λ is a set of edges. In general, the Ising model have been studied on integer lattice $\Gamma=Z^d$, with $d\geq 2$ and on the Cayley tree $\Gamma=(V,\Lambda)$, and for these models the problem of phase transition solved, see for example (Baxter, 1982; Georgii, 1988).

Below, we consider a semi-infinite Cayley tree of second order $\Gamma_+^2=(V,\Lambda)$, i.e., an infinite graph without cycles with 3 edges issuing from each vector except for x^0 which has only 2 edges, where V is the set of its vertices and Λ is the set of edges.

Two vertices $x,y\in V$ are called nearest-neighbors if there exists and edge $l\in\Lambda$ connecting them, which is denoted by $l=\langle x,y\rangle$.

At each vertex or site, there is a small dipole or spin which point at any given moment is in one of two positions, up or down. A configuration σ is defined as function $\sigma: V\to\Phi=\{-1,1\}$, where $\sigma(x)=1$ indicating a spin up and $\sigma(x)=-1$ a spin down. To each configuration σ an energy (Hamiltonian) $H(\sigma)$ is assigned by

$$H(\sigma)=-\sum_{\langle x,y\rangle}J_{xy}\sigma(x)\sigma(y), \quad (1)$$

which represents the energy caused by interaction of the spins. Here, J_{xy} signifies the interaction energy between neighboring spins. Using this model for the case $J_{xy}=J$, Ising tried to explain certain empirically observed facts about ferromagnetic materials. Ising made the simplifying assumption that only interactions between neighboring spins need be considered.

The Potts model (Potts, 1952) was introduced as a generalization for the Ising model to more than two components and encompasses a number of problems in statistical physics (see for example, (Zakharevich, 1978)). The model is structured richly enough to illustrate almost every conceivable nuance of the subject.

For the three-state Potts model with spin values in $\Phi=\{1,2,3\}$, the relevant Hamiltonian with nearest-neighbor interactions has the form

$$H(\sigma)=-\sum_{\langle x,y\rangle}\delta_{\sigma(x)\sigma(y)}, \quad (2)$$

where $J\in R$ is coupling constant and δ is the Kronecker symbol, that is

$$\delta_{\sigma(x)\sigma(y)}=\begin{cases}1 & \text{if } \sigma(x)=\sigma(y)\\ 0 & \text{otherwise.}\end{cases}$$

In Minlos (2000), Minlos introduced the three-component model as follows. Assume $\Phi=\{v_1,v_2,v_3\}$ is the spin system and the Hamiltonian $H_2(\sigma)$ of the configuration σ, then

$$H_2(\sigma)=-J\sum_{\langle x,y\rangle}\varepsilon(\sigma(x)\sigma(y)). \quad (3)$$

Here $\varepsilon(v_i, v_j) = \varepsilon_{ij}$, $i, j = 1,2,3$ is a symmetric matrix A of the third order. If

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

we have the Potts model.

Rock-Paper-Scissors games

During the last decades within game theory, evolutionary and dynamical aspects have exploded. Hofbauer and Sigmund's book, "Evolutionary Games and Population Dynamics" (Hofbauer & Sigmund, 1998) can serve as a very good introduction to this theory. Zero-sum games and their evolutionary dynamics was studied by Akin and Losert (Akin & Losert, 1984) (see also (Hofbauer & Sigmund, 1998) and references therein). We recall the definition of zero-sum games following (Akin & Losert, 1984) and show their connection with the Volterra dynamical systems. A two-player symmetric game consists of a finite set of strategies indexed by $\Phi = \{1,...,m\}$ and an $m \times m$ payoff matrix (a_{ij}). When an i player meets a j player their payoffs are a_{ij} and a_{ji}, respectively. In evolutionary game dynamics, it is supposed a large population is a vector in $\mathbb{R}_+^m = \{p \in \mathbb{R}^m : p_i \geq 0\}$ where p_i measures the subpopulation of i strategists. Therefore, the total population size is $|p| = \sum_i p_i$. The associated distribution vector $\mathrm{x} = (x_1,...,x_m)$ lies in the simplex $S^{m-1} = \left\{ \mathrm{x} = (x_1, x_2,...,x_m) \in R^m : \text{for any } i, x_i \geq 0 \text{ and } \sum_{i=1}^m x_i = 1 \right\}$, where $x_i = \frac{p_i}{|p|}$, is the ratio of i strategists to the total population. Then the nonlinear operator $V : S^{m-1} \to S^{m-1}$ with

$$(V\mathrm{x})_k = x_k \left(1 + \sum_{i=1}^m a_{ki} x_i\right), \tag{4}$$

where $A = (a_{ij})_1^m$ is a skew-symmetric matrix with $|a_{ij}| \leq 1$ can be reinterpreted in terms of evolutionary games and in those forms it has a fair history (Akin & Losert, 1984; Hofbauer & Sigmund, 1998; Losert & Akin, 1983). The Hofbauer and Sigmund's book (Hofbauer & Sigmund, 1998) reflects well the state of evolutionary games till 1998.

Rock-Paper-Scissors is a three strategic 2-player game. According to game rules rock beats scissors, scissors beats paper and paper beats rock. For brevity, we rename these three strategies as follows $1 = Rock$; $2 = Paper$; $3 = Scissors$. Then, corresponding payoff matrix has the form

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

Volterra quadratic stochastic operators

The nonlinear operator $V : S^{m-1} \to S^{m-1}$ with

$$(V\mathrm{x})_k = x_k \left(1 + \sum_{i=1}^m a_{ki} x_i\right), \tag{5}$$

where $A = (a_{ij})_1^m$ is a skew-symmetric matrix with $|a_{ij}| \leq 1$ is called Volterra quadratic stochastic operators (Ganikhodjaev, Ganikhodjaev, & Jamilov, 2015; Ganikhodjaev & Zanin, 2004; Hui & Xu, 2018; Kesten, 1970). Recall ergodic hypothesis for quadratic stochastic operators (Ulam, 1960).

A qso V is called regular if for any initial point $\mathrm{x} \in S^{m-1}$ the limit

$$\lim_{n\to\infty} V\left(\mathrm{x}^{(n)}\right), \tag{6}$$

exists.

A qso V is said to be ergodic if the limit

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} V^k(x), \tag{7}$$

exists for any $\mathrm{x} \in S^{m-1}$.

Note that a regular qso V is ergodic, but in generally from ergodicity does not follow regularity. On the basis of numerical calculations for special type of nonlinear transformations, namely, so-called quadratic stochastic operators (Bernstein, 1924), Ulam conjectured that the ergodic theorem holds for any quadratic stochastic operator V (Ulam, 1960). In 1977 Zakharevich proved that this conjecture is false in general (Zakharevich, 1978). He proved that the following quadratic stochastic operator

$$\begin{aligned} x_1' &= x_1^2 + 2x_1x_2, \\ x_2' &= x_2^2 + 2x_2x_3, \\ x_3' &= x_3^2 + 2x_1x_3, \end{aligned} \tag{8}$$

generated by the matrix

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \tag{9}$$

is non-ergodic transformation.

As evolutionary games are crucial in studying the emergence of cooperation in a competing population (Hofbauer & Sigmund, 1998) and the Ising model is one of the best studied models in statistical physics (Christensen & Moloney, 2015), a connection between them would allow two fields to borrow ideas, techniques and results from each other. In Hui and Xu (2018), and Liu, Xu and Hui (2017) the authors established such mapping via arguments based on defailed balance. More on previous works on relating the games and opinion formation model to the Ising model can be found in Liu et al. (2017).

In this paper, we discuss a connection between Paper-Rock-Scissors game, three component Potts model, and non-ergodic Volterra quadratic stochastic operators.

ROCK-PAPER-SCISSORS MODEL

Let $\Phi = \{1,2,3\}$ be the set of spins and Γ_+^2 be a semi-infinite Cayley tree of order 2, i.e. infinite graph without cycles with 3 edges issuing from each vertex except for x^0 which has only 2 edges. The distance $d(x,y), x,y \in V$ on Γ_+^2, is the number of edges in the shortest path from x to y. For a fixed $x^0 \in V$ we set

$$V_n = \left\{ x \in V \middle| d(x, x^0) \leq n \right\},$$

and L_n denotes the set of edges in V_n. We introduce RPS-model on Γ_+^2 with the following Hamiltonian

$$H(\sigma) = -J \sum_{\langle x,y\rangle \in L} a_{\sigma(x)\sigma(y)}, \tag{10}$$

where $J \in R$ is coupling constant, $a_{ij} \in A$ (9) and $\langle x,y\rangle$ stands for nearest-neighbor vertices. Note that the matrix A (9) is skew-symmetric, such that the considered model is differ than three-component model from Minlos (2000).

In order to produce the recurrent equations we consider the relation of the partition on finite subset $V_n \subset V$ to the partition function on subset of V_{n-1}, where $V_1 \subset V_2 \subset ... \subset V_n \subset V_{n+1} \subset ...$ a sequence of finite subsets such that $\bigcup_{n=1}^{\infty} V_n = V$. Let $Z_i^{(n)}$ be a partition function on V_n with the spin i in the root x^0, $i = 1,2,3$. Let $a = \exp(\beta J)$. Following Kindermann and Snell (1980), one can produce the following recurrent equation

$$\begin{aligned} Z_1^{(n+1)} &= (Z_1^{(n)} + aZ_2^{(n)} + a^{-1}Z_3^{(n)})^2, \\ Z_2^{(n+1)} &= (a^{-1}Z_1^{(n)} + Z_2^{(n)} + aZ_3^{(n)})^2, \\ Z_3^{(n+1)} &= (aZ_1^{(n)} + a^{-1}Z_2^{(n)} + Z_3^{(n)})^2. \end{aligned} \tag{11}$$

Denote $u^{(n)} = \dfrac{Z_1^{(n)}}{Z_3^{(n)}},\ v^{(n)} = \dfrac{Z_2^{(n)}}{Z_3^{(n)}}$.

Then, according to (11) we have the following dynamical system

$$\begin{aligned} u^{(n+1)} &= \left(\frac{au^{(n)} + a^2 v^{(n)} + 1}{a^2 u^{(n)} + v^{(n)} + a} \right)^2, \\ v^{(n+1)} &= \left(\frac{u^{(n)} + av^{(n)} + a^2}{a^2 u^{(n)} + v^{(n)} + a} \right)^2. \end{aligned} \tag{12}$$

It is evident that behavior of the trajectory of this dynamical system describes limit Gibbs measure of the considered model, namely, attractive fixed point describes translation-invariant Gibbs measure and repelling fixed points describes periodic Gibbs measures. Below we investigate fixed points of dynamical system (12). Consider the following system of equations

$$\begin{aligned} u &= \left(\frac{au + a^2 v + 1}{a^2 u + v + a} \right)^2, \\ v &= \left(\frac{u + av + a^2}{a^2 u + v + a} \right)^2, \end{aligned} \tag{13}$$

where $u = \lim_{n\to\infty} u^{(n)}$ and $v = \lim_{n\to\infty} v^{(n)}$.

It is evident that $u = 1$; $v = 1$ is the fixed point. Let us compute Jacobian

$$J_{u,v}(1,1) = \begin{bmatrix} -\dfrac{2a(a-1)}{1+a+a^2} - \lambda & \dfrac{2(a^2-1)}{1+a+a^2} \\ -\dfrac{2(a^2-1)}{1+a+a^2} & \dfrac{2(a-1)}{1+a+a^2} - \lambda \end{bmatrix}.$$

Then, corresponding characteristic polynomial has the following form

$$\lambda^2 + \frac{2(a-1)^2}{1+a+a^2}\lambda + \frac{4(a-1)^2}{1+a+a^2} = 0, \tag{14}$$

with discriminant

$$\Delta = -\frac{12(a-1)^2(a+1)^2}{(1+a+a^2)^2}. \tag{15}$$

Thus, the eigenvalues are

$$\lambda_{1,2} = \frac{-(a-1)^2 \pm i\sqrt{3}(a+1)\sqrt{(a-1)^2}}{(1+a+a^2)}, \tag{16}$$

with

$$|\lambda|^2 = \frac{4(a-1)^2}{(1+a+a^2)}. \tag{17}$$

Then $|\lambda| > 1$ if $a < \dfrac{3-\sqrt{5}}{2} \approx 0.382$ or $a > \dfrac{3+\sqrt{5}}{2} \approx 2.618$. Therefore, we have $|\lambda| < 1$ for $a \in (0.382; 2.618)$ and respectively the fixed point $(1,1)$ is attractive, and if $a < 0.382$ or $a > 2.618$ then $|\lambda| > 1$ and respectively the fixed point is repelling.

Applying numerical methods, one can show that for $a < 0.382$ or $a > 2.618$, there exists a cycle of third order and all trajectories tend to this cycle except the stationary trajectory starting with fixed point.

Thus, the main result of this paper is formulated as follows:

Theorem: The phase diagram of Rock-Paper-Scissors model on Cayley tree of second order consists of translation-invariant Gibbs measure if $\beta|J| < 0.9624$ and periodic Gibbs measure with period 3 if $\beta J < -0.9624$ or $\beta J > 0.9624$.

CONCLUSION

It is shown that the matrix A (1.9) can be reinterpreted in terms of evolutionary games, namely Rock-Paper-Scissors game and also can be reinterpreted in terms of non-ergodic Volterra quadratic stochastic operator (1.8). It is introduced Rock-Paper-Scissors model on Cayley tree of second order generated by the same matrix A and described its phase diagram.

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