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A comparison of multivariate control charts for skewed distributions using weighted standard deviations

Michael B.C. Khoo*, Sin Yin Teh, May Yin Eng

School of Mathematical Sciences, Universiti Sains Malaysia, 11800 Minden, Penang, Malaysia.

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ABSTRACT

The quality of a manufacturing process usually depends on more than one quality characteristic. Thus, most process monitoring data are multivariate in nature. The assumption that the underlying process follows a multivariate normal distribution is usually required by most multivariate quality control charts. However, in most process monitoring situations, the multivariate normality assumption is often violated. Multivariate control charts for skewed distributions have been suggested to enable process monitoring to be made when the underlying process distribution is skewed. Among the recent heuristic multivariate charts for skewed distributions suggested in the literature are those based on the weighted standard deviation (WSD) approach. This paper compares the performances of three multivariate charts for skewed distributions incorporating the WSD method, namely, the WSD T^2, WSD multivariate cumulative sum (WSD MCUSUM) and WSD multivariate exponentially weighted moving average (WSD MEWMA) charts. These heuristic charts are compared based on the multivariate lognormal, gamma and Weibull distributions. The charts' performances are evaluated using the false alarm rates, computed via a Monte-carlo simulation. The chart with the lowest false alarm rate for most of the skewness levels and sample sizes will be identified as the chart having the best performance.

| Multivariate control charts | Skewed distributions | Weighted standard deviations | WSD T^2 | WSD MEWMA| WSD MCUSUM |

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1. INTRODUCTION

In process monitoring, we often deal with two or more related variables [1]. An individual might have a false understanding that by applying univariate control charts to each of the variables, instead of a multivariate chart, the same results can be obtained. Previous studies have shown that using separate univariate charts are not only inefficient and time consuming, but they also lead to misleading or erroneous conclusions. Multivariate statistical quality control charts which consider different variables simultaneously are required [2].

Three common types of multivariate control charts are the Hotelling's T^2, multivariate exponentially weighted moving average (MEWMA) and multivariate cumulative sum (MCUSUM) charts. The Hotelling's T^2 chart is the most common multivariate process monitoring procedure in controlling the mean vector of a process. This control chart is actually an analog of the univariate Shewhart mean chart [2]. The MEWMA chart has been developed to overcome the disadvantages of the Hotelling's T^2 chart, such as its insensitivity to small and moderate shifts. A MEWMA chart is a logical extension of the univariate EWMA chart. Like MEWMA, the MCUSUM chart is also a useful chart in detecting small shifts. There are two versions of the MCUSUM charts, namely the MCUSUM#1 ($MC1$) chart and the MCUSUM#2 ($MC2$) chart. Pignatiello and Runger [2] stated that comparatively for both $MC1$ and $MC2$ charts, the $MC1$ chart seems to be more efficient in detecting small shifts than the $MC2$ chart. As for the $MC2$ chart, it appears to be more efficient in detecting large shifts compared to the $MC1$ chart.

The multivariate charts that are mentioned above are confined to multivariate normal populations. However, the normality assumption is found to be difficult to justify and is often violated [3].

Chang and Bai [3] proposed a heuristic method for constructing the multivariate T^2 control chart for skewed populations. Chang [4] also suggested a heuristic method of constructing the MEWMA and MCUSUM charts for skewed distributions.

The layout of this paper is as follows: Section 2 explains the weighted standard deviation method, Sections 3, 4 and 5 discuss the WSD T^2 chart, WSD MEWMA chart and WSD MCUSUM chart, respectively. Section 6 compares the performances of the three types of multivariate charts for skewed distributions. Conclusions are drawn in Section 7.

Corresponding author at: *School of Mathematical Sciences Universiti Sains Malaysia, 11800 Minden, Penang, Malaysia.*
E-mail addresses: mkbc@usm.my (Michael B.C. Khoo)

2. WEIGHTED STANDARD DEVIATION (WSD) METHOD

Chang and Bai [3] proposed the WSD method for constructing the multivariate T^2 control chart for skewed populations. The WSD method approximates the probability density function (pdf) of a v-variate skewed distribution with segments from 2^v multivariate normal distributions by modifying the variance-covariance matrix according to the estimated degree and direction of the skewness.

The proposed WSD method assumes that a v-variate random vector $\boldsymbol{X} = (X_1, ..., X_v)^T$ is distributed with a multivariate skewed distribution having a mean vector

$$\boldsymbol{\mu} = (\mu_1, ..., \mu_v)^T \tag{1}$$

and variance-covariance matrix,

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \cdots & \rho_{1v}\sigma_1\sigma_v \\ & \sigma_2^2 & \cdots & \rho_{2v}\sigma_2\sigma_v \\ & & \ddots & \vdots \\ & & & \sigma_v^2 \end{bmatrix}, \tag{2}$$

where σ_j is the standard deviation of X_j and ρ_{ij} is the correlation coefficient of X_i and X_j.

The WSD method works by adjusting the variance-covariance matrix. Hence, the variance-covariance matrix of the 2^v multivariate normal distributions for the approximation is [3]

$$\boldsymbol{\Sigma}^{\mathrm{W}} = \mathbf{W}\cdot\boldsymbol{\Sigma}\cdot\mathbf{W} = \begin{bmatrix} \left(\sigma_1^W\right)^2 & \rho_{12}\sigma_1^W\sigma_2^W & \cdots & \rho_{1v}\sigma_1^W\sigma_v^W \\ & \left(\sigma_2^W\right)^2 & \cdots & \rho_{2v}\sigma_2^W\sigma_v^W \\ & & \ddots & \vdots \\ & & & \left(\sigma_v^W\right)^2 \end{bmatrix}, \tag{3}$$

where $\boldsymbol{W} = \operatorname{diag}\left\{W_1(X_1 - \mu_1), ..., W_v(X_v - \mu_v)\right\}$, and

$$W_j(x) = \begin{cases} 2P_j & \text{if } x > 0 \\ 2(1 - P_j) & \text{otherwise} \end{cases} \tag{4}$$

Note that $\sigma_j^W = W_j(X_j - \mu_j)\cdot\sigma_j$ and $P_j = \Pr(X_j \le \mu_j)$. This means that if a skewed distribution is found to be skewed to the right, then $P_j > \frac{1}{2}$ and $2P_j\sigma_j > 2(1 - P_j)\sigma_j$. However, the correlation matrix $\boldsymbol{\rho} = \{\rho_{ij}\}$ does not change.

3. THE WSD T^2 CHART

Chang and Bai [3] proposed the T^2 statistic based on the WSD method by defining $\boldsymbol{Z}^W = \boldsymbol{W}^{-1}\boldsymbol{Z}$, where the j^{th} element of $\boldsymbol{Z}^W$ is

$$Z_j^W = \frac{X_j - \mu_j}{\sigma_j^W} = \frac{X_j - \mu_j}{W_j\sigma_j} = \begin{cases} \dfrac{1}{2P_j}Z_j & \text{if } X_j > \mu_j \\ \dfrac{1}{2(1 - P_j)}Z_j & \text{, otherwise} \end{cases}. \tag{5}$$

Then the WSD T^2 statistic is defined as

$$T_{W,i}^2 = \left(n\bar{\boldsymbol{Z}}_i^W\right)^T \rho^{-1}\bar{\boldsymbol{Z}}_i^W, \tag{6}$$

where $\bar{\boldsymbol{Z}}_i^W = \left(\bar{Z}_{1i}^W, ..., \bar{Z}_{vi}^W\right)^T$. An out-of-control signal is detected when $T_{W,i}^2 > \chi_\alpha^2(v)$, where $\chi_\alpha^2(v)$ is defined as the $100(1-\alpha)^{\text{th}}$ percentile of the χ^2 distribution with v degree of freedom.

4. THE WSD MEWMA CHART

The performance of the MEWMA chart for skewed populations can be improved by applying the WSD method. The WSD MEWMA statistic is defined as [4]:

$$\boldsymbol{M}_i^W = \lambda\boldsymbol{Z}_i^W + (1 - \lambda)\boldsymbol{M}_{i-1}^W, \tag{7}$$

where i = 1, 2, … and $\boldsymbol{M}_0^W = \mathbf{0}$.

Since $\boldsymbol{Z}_i^W$ follows a multivariate normal distribution with mean vector $\mathbf{0}$ and variance-covariance matrix, $\boldsymbol{\rho}$ approximately, then the asymptotic variance-covariance matrix of $\boldsymbol{M}_i^W$ is

$$\boldsymbol{\Sigma}_{\boldsymbol{M}} = \left[\frac{\lambda}{(2 - \lambda)}\right]\boldsymbol{\rho}, \tag{8}$$

which is the same as that of the standard MEWMA statistic. Hence, when the charting statistic exceeds h_E, that is

$$E_i^W = \left(\boldsymbol{M}_i^W\right)^T \Sigma_M^{-1}\boldsymbol{M}_i^W > h_E, \tag{9}$$

the WSD MEWMA chart issues an out-of-control signal. When the distribution is symmetric, the proposed WSD MEWMA chart reduces to the standard MEWMA chart [4].

5. THE WSD MCUSUM CHART

The WSD MCUSUM chart modifies the MCUSUM charting statistic with the degree and direction of the skewness. Based on the WSD method, the sample observations are first standardized as $\boldsymbol{Z}^W = \boldsymbol{W}^{-1}\boldsymbol{Z}$, of which the *j*th element is defined as [4]

$$Z_j^W = \frac{X_j - \mu_j}{\sigma_j^W} = \frac{X_j - \mu_j}{W_j \sigma_j} = \begin{cases} \dfrac{1}{2P_j} Z_j & \text{if } X_j > \mu_j \\ \dfrac{1}{2\left(1-P_j\right)} Z_j & \text{, otherwise} \end{cases} \tag{10}$$

Note that $\boldsymbol{Z}^W$ follows a multivariate normal distribution, approximately. The cumulative statistic at *i* is defined as

$$\boldsymbol{A}_i^W = \boldsymbol{S}_{i-1}^W + \boldsymbol{Z}_i^W , \tag{11}$$

where the length of $\boldsymbol{A}_i^W$ is $C_i^W = \left[\left(\boldsymbol{A}_i^W\right)^T \rho^{-1} \boldsymbol{A}_i^W\right]^{1/2}$, and $\boldsymbol{S}_i^W$ is the cumulative sum at *i*. The size of the mean shift $d\left(\mu_1\right)$ is also modified according to the adjusted variance-covariance matrix, using the WSD method as follows:

$$d^W\left(\mu_1\right) = \left[\left(\mu_1 - \mu\right)^T \left(\Sigma^W\right)^{-1} \left(\mu_1 - \mu\right)\right]^{1/2} . \tag{12}$$

Let $\boldsymbol{\Sigma}^W = \boldsymbol{W} \cdot \boldsymbol{\Sigma} \cdot \boldsymbol{W}$ (see Equation (3)) and $k^W = \dfrac{d^W\left(\mu_1\right)}{2}$ be the reference value. Chang [4] mentioned that when $C_i^W > k^W$, the cumulative sum will shrink towards **0**. He pointed out that the same direction and size cannot be applied to all the variables because the upper and lower WSDs of each of the variables are not the same. Hence, the j^{th} element of the cumulative sum $\boldsymbol{S}_i^W$ is defined as [4]

$$S_{ji}^W = \begin{cases} A_{ji}^W \left(1 - \dfrac{k}{2P_j C_i^W}\right) & \text{if } C_i^W > k^W \text{ and } A_{ji}^W > 0 \\ A_{ji}^W \left(1 - \dfrac{k}{2\left(1-P_j\right) C_i^W}\right) & \text{if } C_i^W > k^W \text{ and } A_{ji}^W \le 0 \\ 0 & \text{otherwise} \end{cases} . \tag{13}$$

If $Y_i^W = \left[\left(\boldsymbol{S}_i^W\right)^T \rho^{-1} \boldsymbol{S}_i^W\right]^{1/2}$ exceeds h_C, the WSD MCUSUM chart will issue an out-of-control alarm. Here, h_C is determined based on a desired in-control ARL. When the distribution is symmetric, the proposed WSD MCUSUM chart reduces to the MCUSUM chart [4].

6. A COMPARISON OF THE PERFORMANCES OF THREE TYPES OF MULTIVARIATE CHARTS FOR SKEWED DISTRIBUTIONS

This section compares the performances of three different types of multivariate control charts for skewed distributions, namely the WSD T^2, WSD MEWMA and WSD MCUSUM charts. The skewed distributions considered are the lognormal, gamma and Weibull distributions. The *Statistical Analysis Syste*m (*SAS*) version 9 is used to conduct the simulation studies of the charts based on the bivariate case, which involves two quality characteristics (p = 2).

In general, all the three bivariate WSD charts for skewed distributions are compared based on the skewness coefficients of $\left(\gamma_1, \gamma_2\right) = (1,1), (1,2), (1,3), (2,2), (2,3)$ and $(3,3)$ and correlation coefficients of ρ = 0.3, 0.5 and 0.8. The sample size, *n*, for all the three WSD charts are fixed as *n* = 1, 5 and 10.

From the simulation results in Tables 1 – 3, it is found that the false alarm rates for all the three types of the bivariate WSD charts, based on the bivariate lognormal, gamma and Weibull distributions increase as the level of skewness increases. For example, the false alarm rate for $\left(\gamma_1, \gamma_2\right) = (2,2)$ is higher than that of $\left(\gamma_1, \gamma_2\right) = (1,1)$. Note that for all three types of the bivariate WSD charts, based on gamma distribution (see Tables 1 – 3), the symbol "*" is used to represent the false alarm rate that cannot be computed by SAS as there exists a negative value in the shape parameter of the gamma distribution.

The results in Table 1 shows that the false alarm rate of the WSD T^2 chart decreases as the sample size increases. In contrast, the results in Tables 2 and 3 show that, generally, the false alarm rates of the WSD MEWMA and WSD MCUSUM charts increase with the sample size.

In general, the WSD MCUSUM chart has the lowest false alarm rate while the WSD T^2 chart has the highest, when the same level of skewness, sample size and underlying distribution are considered.

Table 1: False alarm rates for the WSD T^2 chart, based on the bivariate lognormal, gamma and Weibull distributions for ρ ∈ {0.3, 0.5, 0.8} and n ∈ {1, 5, 10}

ρ	*n*	(γ_1,γ_2)	Distribution: lognormal	gamma	Weibull
0.3	1	(1,1)	0.008700	0.008185	0.007923
		(1,2)	0.010960	0.010507	0.010959
		(1,3)	0.011739	0.011331	0.012168
		(2,2)	0.013349	0.013173	0.013951
		(2,3)	0.014127	0.014343	0.015145
		(3,3)	0.014899	0.015441	0.016313
	5	(1,1)	0.003317	0.003122	0.002835
		(1,2)	0.004250	0.003459	0.003576
		(1,3)	0.005221	0.004100	0.004567
		(2,2)	0.005278	0.004210	0.004330
		(2,3)	0.006267	0.004840	0.005392
		(3,3)	0.007314	0.005696	0.006623
	10	(1,1)	0.002992	0.002832	0.002749
		(1,2)	0.003376	0.002992	0.003174
		(1,3)	0.003994	0.003762	0.003778
		(2,2)	0.003834	0.003184	0.003590
		(2,3)	0.004487	0.003841	0.004240
		(3,3)	0.005129	0.004274	0.004884
0.5	1	(1,1)	0.008949	0.009151	0.011196
		(1,2)	0.011290	0.010907	0.014008
		(1,3)	0.011868	*	0.014971
		(2,2)	0.013750	0.014903	0.016789
		(2,3)	0.014400	0.015704	0.017804
		(3,3)	0.015158	0.016691	0.018735
	5	(1,1)	0.003345	0.003398	0.003444
		(1,2)	0.004376	0.003963	0.004472
		(1,3)	0.005391	*	0.005900
		(2,2)	0.005459	0.005159	0.005667
		(2,3)	0.006499	0.006036	0.007189
		(3,3)	0.007531	0.007221	0.008770
	10	(1,1)	0.002986	0.002985	0.002934
		(1,2)	0.003381	0.003579	0.003569
		(1,3)	0.004034	*	0.004668
		(2,2)	0.003915	0.004026	0.004308
		(2,3)	0.004556	0.005182	0.005470
		(3,3)	0.005271	0.005997	0.006650
0.8	1	(1,1)	0.009382	0.013133	0.022012
		(1,2)	0.011369	*	0.023218
		(1,3)	0.011167	*	0.023560
		(2,2)	0.014180	0.020178	0.023732
		(2,3)	0.014583	*	0.023662
		(3,3)	0.015405	0.021604	0.023395
	5	(1,1)	0.003489	0.004908	0.007392
		(1,2)	0.004689	*	0.010716
		(1,3)	0.005761	*	0.014355
		(2,2)	0.005835	0.020178	0.013566
		(2,3)	0.006929	*	0.016718
		(3,3)	0.008033	0.021604	0.019283
	10	(1,1)	0.003058	0.004048	0.004894
		(1,2)	0.003580	*	0.007581
		(1,3)	0.004417	*	0.011791
		(2,2)	0.004094	0.008394	0.009848
		(2,3)	0.004917	*	0.013451
		(3,3)	0.005676	0.015986	0.016788

Table 2: False alarm rates for the WSD MEWMA chart, based on the bivariate lognormal, gamma and Weibull distributions for ρ ∈ {0.3, 0.5, 0.8} and n ∈ {1, 5, 10}

ρ	*n*	(γ_1,γ_2)	Distribution: lognormal	gamma	Weibull
0.3	1	(1,1)	0.003042	0.003009	0.003215
		(1,2)	0.003434	0.004167	0.004212
		(1,3)	0.003700	0.006534	0.005158
		(2,2)	0.0036990	0.004740	0.004967
		(2,3)	0.003925	0.006530	0.005877
		(3,3)	0.004098	0.008636	0.006882
	5	(1,1)	0.004071	0.004202	0.004419
		(1,2)	0.005698	0.007949	0.007847
		(1,3)	0.007511	0.015924	0.012554
		(2,2)	0.00711	0.010586	0.010526
		(2,3)	0.008732	0.01773	0.014738
		(3,3)	0.010366	0.023721	0.018382
	10	(1,1)	0.004340	0.004594	0.004820
		(1,2)	0.006409	0.008839	0.008820
		(1,3)	0.008862	0.018181	0.014634
		(2,2)	0.008114	0.012147	0.011973
		(2,3)	0.010297	0.010446	0.017103
		(3,3)	0.012359	0.027005	0.021457
0.5	1	(1,1)	0.002909	0.002886	0.002988
		(1,2)	0.003264	0.004288	0.003931
		(1,3)	0.003518	*	0.005133
		(2,2)	0.003256	0.004092	0.004303
		(2,3)	0.003355	0.006037	0.005111
		(3,3)	0.003413	0.006870	0.005702
	5	(1,1)	0.003853	0.003989	0.003977
		(1,2)	0.005425	0.007994	0.007359
		(1,3)	0.007305	*	0.012743
		(2,2)	0.006356	0.009752	0.009478
		(2,3)	0.007796	0.016576	0.013414
		(3,3)	0.009052	0.021121	0.016431
	10	(1,1)	0.004158	0.004436	0.004281
		(1,2)	0.006215	0.009018	0.008543
		(1,3)	0.008715	*	0.014895
		(2,2)	0.007420	0.010989	0.010870
		(2,3)	0.009406	0.019249	0.015809
		(3,3)	0.010991	0.024673	0.019531
0.8	1	(1,1)	0.002773	0.003283	0.003464
		(1,2)	0.003240	*	0.005210
		(1,3)	0.003787	*	0.008147
		(2,2)	0.002826	0.004385	0.004642
		(2,3)	0.002878	*	0.005537
		(3,3)	0.002755	0.006216	0.005498
	5	(1,1)	0.003647	0.004062	0.003579
		(1,2)	0.005659	*	0.008457
		(1,3)	0.008696	*	0.018611
		(2,2)	0.005619	0.009255	0.008604
		(2,3)	0.007042	*	0.013355
		(3,3)	0.007583	0.019161	0.014879
	10	(1,1)	0.003971	0.004240	0.003739
		(1,2)	0.006592	*	0.009785
		(1,3)	0.010563	*	0.021683
		(2,2)	0.006646	0.010338	0.009895
		(2,3)	0.008785	*	0.015942
		(3,3)	0.009539	0.021977	0.017833

Table 3: False alarm rates for the WSD MCUSUM chart, based on the bivariate lognormal, gamma and Weibull distributions for $\rho \in \{0.3, 0.5, 0.8\}$ and $n \in \{1, 5, 10\}$

ρ	*n*	(γ_1, γ_2)	Distribution		
			lognormal	**gamma**	**Weibull**
0.3	1	(1,1)	0.002567	0.002527	0.002264
		(1,2)	0.002557	0.002266	0.002219
		(1,3)	0.002617	0.002188	0.002304
		(2,2)	0.002563	0.002088	0.002130
		(2,3)	0.002692	0.002114	0.002274
		(3,3)	0.002802	0.002073	0.002479
	5	(1,1)	0.002812	0.002793	0.002725
		(1,2)	0.002859	0.003089	0.003026
		(1,3)	0.002858	0.003388	0.003167
		(2,2)	0.002791	0.003092	0.003191
		(2,3)	0.002768	0.003392	0.003223
		(3,3)	0.002721	0.003501	0.003236
	10	(1,1)	0.002981	0.002967	0.002880
		(1,2)	0.003203	0.003590	0.003549
		(1,3)	0.003366	0.004765	0.004127
		(2,2)	0.003355	0.003971	0.003983
		(2,3)	0.003508	0.004973	0.004473
		(3,3)	0.003598	0.005735	0.004858
0.5	1	(1,1)	0.002572	0.002467	0.002369
		(1,2)	0.002599	0.002325	0.002455
		(1,3)	0.002620	*	0.002750
		(2,2)	0.002629	0.002319	0.002491
		(2,3)	0.002744	0.002392	0.003071
		(3,3)	0.002922	0.002370	0.003069
	5	(1,1)	0.002752	0.002969	0.002446
		(1,2)	0.002780	0.003138	0.002851
		(1,3)	0.002770	*	0.003189
		(2,2)	0.002672	0.003012	0.002934
		(2,3)	0.002605	0.003233	0.003071
		(3,3)	0.002495	0.003069	0.003069
	10	(1,1)	0.002903	0.002977	0.002627
		(1,2)	0.003105	0.003634	0.003343
		(1,3)	0.003237	*	0.004156
		(2,2)	0.003150	0.003803	0.003660
		(2,3)	0.003215	0.004616	0.004167
		(3,3)	0.003231	0.005043	0.004478
0.8	1	(1,1)	0.002563	0.002908	0.003248
		(1,2)	0.002683	*	0.003811
		(1,3)	0.002877	*	0.004795
		(2,2)	0.002769	0.003107	0.003887
		(2,3)	0.002963	*	0.004337
		(3,3)	0.003186	0.003445	0.004601
	5	(1,1)	0.002711	0.002786	0.002361
		(1,2)	0.002818	*	0.003385
		(1,3)	0.002921	*	0.005311
		(2,2)	0.002549	0.003136	0.003153
		(2,3)	0.002459	*	0.003725
		(3,3)	0.002373	0.003378	0.003725
	10	(1,1)	0.002827	0.002989	0.002277
		(1,2)	0.003165	*	0.003832
		(1,3)	0.003483	*	0.006800
		(2,2)	0.002976	0.003756	0.003521
		(2,3)	0.003026	*	0.004559
		(3,3)	0.002910	0.004942	0.004712

7. CONCLUSION

In statistical quality control, the assumption that the underlying distribution is normal is usually made. However, this leads to erroneous conclusion if the normality assumption is violated. In practice, many processes come from skewed populations with multivariable or quality characteristics. In this paper, three multivariate control charts for skewed populations, namely the Hotelling's T^2, MCUSUM and MEWMA charts using the weighted standard deviation (WSD) method, based on the bivariate lognormal, gamma and Weibull distributions are considered. The false alarm rates of the charts computed using the SAS program are compared, for various levels of skewness and sample sizes. The WSD MCUSUM chart is found to have the lowest false alarm rate for the various levels of skewness considered. This study helps practitioners in deciding the type of chart to be used in process monitoring.

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