Malaysian Journal of Fundamental & Applied Sciences Vol.8, No.4 (2012) 193-197

Malaysian Journal of Fundamental & Applied Sciences

available online at http://mjfas.ibnusina.utm.my

Solving a Mixed Boundary Value Problem via an Integral Equation with Generalized Neumann Kernel on Unbounded Multiply Connected Region

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Received 26 November 2011, Revised 1 May 2012, Accepted 22 May 2012, Available online 15 June 2012

ABSTRACT

In this paper, we solve the mixed boundary value problem on unbounded multiply connected region by using the method of boundary integral equation. Our approach in this paper is to reformulate the mixed boundary value problem into the form of Riemann-Hilbert problem. The Riemann-Hilbert problem is then solved using a uniquely solvable Fredholm integral equation on the boundary of the region. The kernel of this integral equation is the generalized Neumann kernel. As an examination of the proposed method, some numerical examples for some different test regions are presented.

| Riemann-Hilbert Problem | Integral Equation | Generalized Neumann Kernel | Laplace Equation | Mixed Boundary Value Problem |

http://dx.doi.org/10.11113/mjfas.v8n4.147

1. INTRODUCTION

The need to solve Laplace equation with different types of boundary conditions on different parts of a connected boundary often arises in computational physics and mechanics. Common mixed boundary conditions are mixed Dirichlet and Neumann type conditions. Recently, the interplay of Riemann-Hilbert problem and integral equation with the generalized Neumann kernel has been investigated in [1-3].

It has been shown that the problem of conformal mapping, Dirichlet problem, and Neumann problem can all be treated as Riemann Hilbert problems [2-4]. Hence they can be solved efficiently using integral equations with the generalized Neumann kernel. The boundary integral equation method is a cl assical method for solving the Dirichlet and Neumann bo undary value problem. The classical boundary integral method for the Dirichlet problem and the Neumann problem are in the form of second kind Fredholm integral equations with the Neumann kernel. Those integral equations are derived by representing the solutions of the mixed problem as the potential of a single layer [5].

In this paper, we extend the result in [6] to solve Laplace equation on unbounded multiply connected region with Dirichlet-Neumann condition via an integral equation with the generalized Neumann kernel. This extends the results of [4]. A Fredholm integral equation of the second kind with the generalized Neumann kernel is derived for the mixed boundary value problem.

This paper is organized as follows: Section 2 presents some auxiliary materials related to the mixed problem, the Riemann-Hilbert problems as well as integral equation for Riemann-Hilbert problems. In Section 3, we reduce the mixed boundary value problem into the Riemann-Hilbert problem and construct the boundary integral equation for solving it. We will discuss the question on how to treat the integral equations numerically in Section 4. Some numerical examples are presented in Section 5. In Section 6, a short conclusion is given.

2. AUXILIARY MATERIAL

Let G be a bounded multiply connected region. The boundary $\Gamma = \partial G$ consists of m Jordan curves $\Gamma_j, j = 1,\ldots,m$ such that the curves Γ_j has clockwise orientation (see Figure 1). The boundaries Γ_j are The parameterization of the whole boundary Γ as the complex function η *is* defined on J by

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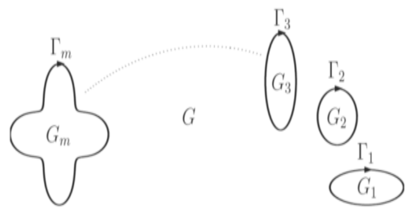

Fig. 1 Unbounded multiply connected region of connectivity *m*

$$\eta(s)=\begin{cases}\eta_1(s), & s\in J_1=[0,2\pi],\\ \eta_2(s), & s\in J_2=[0,2\pi],\\ \vdots & \\ \eta_m(s), & s\in J_m=[0,2\pi].\end{cases} \quad (1)$$

where η_j is twice continuously differentiable with $d\eta/ds\neq 0$. The total parameter region J is the disjoint union of the intervals $J_j, j=1,2,...,m$. Let A_k be a 2π -periodic continuously differentiable function on J_k with $A_k\neq 0$, defined on the total parameterized region J by

$$A(s)=\begin{cases}A_1(s), & s\in J_1=[0,2\pi],\\ A_2(s), & s\in J_2=[0,2\pi],\\ \vdots & \\ A_m(s), & s\in J_m=[0,2].\end{cases} \quad (2)$$

3. GENERALIZED NEUMANN KERNEL AND MIXED BOUNDARY VALUE PROBLEM

In view of the parameterization domain of the boundary Γ, the above formula defines also the function A implicitly on the boundary Γ. We define the reals generalized Neumann kernel *N* by [1-3]

$$N(s,t)=\frac{1}{\pi}\mathrm{Im}\left(\frac{A(s)}{A(t)}\frac{\dot\eta(t)}{\eta(t)-\eta(s)}\right), \quad s\neq t. \quad (3)$$

It is continuous at *s=t* with

$$N(t,t)=\frac{1}{\pi}\mathrm{Im}\left(\frac{1}{2}\frac{\ddot\eta(t)}{\dot\eta(t)}-\frac{\dot A(t)}{A(t)}\right). \quad (4)$$

We define the real kernel M as

$$M(s,t)=\frac{1}{\pi}\mathrm{Re}\left(\frac{A(s)}{A(t)}\frac{\dot\eta(t)}{\eta(t)-\eta(s)}\right), \quad s\neq t. \quad (5)$$

When $s,t\in J_k$ in the same parameter interval J_k,

$$M(s,t)=\frac{1}{2\pi}\cot\left(\frac{s-t}{2}\right)+M_1(s,t), \quad s,t\in J_k. \quad (6)$$

with a continuous kernel M_1 which take on the diagonal the values

$$M(t,t)=\frac{1}{\pi}\Re\left(\frac{1}{2}\frac{\ddot\eta(t)}{\dot\eta(t)}-\frac{\dot A(t)}{A(t)}\right), \quad s=t. \quad (7)$$

Define the integral operators

$$(\mathbf{N}\mu)(s)=\int_J N(s,t)\mu(t)dt, \quad (8)$$

$$(\mathbf{M}\mu)(s)=\int_J M(s,t)\mu(t)dt, \quad (9)$$

where the integral in (9) is a principal value integral.

The solvability of boundary integral equations with the generalized Neumann kernel is determined by the index (winding number in other terminology) of the function *A* [1,5,6].

For the function *A* given by

$$A(s)=\begin{cases}c_1, & s\in J_1=[0,2\pi],\\ c_2, & s\in J_2=[0,2\pi],\\ \vdots & \\ \vdots & \\ c_m, & s\in J_m=[0,2\pi],\end{cases}$$

where c_j are complex constants, the index $\kappa_j, j=1,2,\dots,m$ of A on the curve Γ_j, $j=1,2,\dots,m$ and the indexes

$$\kappa=\sum_{j=1}^{m}\kappa_j,$$

of A of the whole boundary are given by

$$\kappa_J=0, \qquad \kappa=0.$$

Let H be the space of all continuous Hölder functions on the boundary Γ and let S be the subspace of Hwhich consists of all piecewise constant functions defined on Γ. Thus, we have from [1,2,7] the following theorem.

Theorem 2.1. *For a function* $\gamma\in H$, *there exist unique functions* $h\in S$ *and* $\mu\in H$ *such that*

$$Af=\gamma+h+i\mu,$$

are boundary values of unique function $f(z)$*in* G *with* $f(\infty)=0$ *for unbounded* G*, where* μ *is a unique solution of the integral equation*

$$\mu - \boldsymbol{N}\mu = -\boldsymbol{M}\gamma,$$

and the function h is given by

$$h = [\boldsymbol{M}\mu - (\boldsymbol{I}-\boldsymbol{N})\gamma]/2.$$

4. REDUCTION OF MIXED BOUNDARY VALUE PROBLEM TO RIEMANN -HILBERT PROBLEM.

In this section we show how to reduce the mixed boundary problem into the form of Riemann-Hilbert problem on multiply connected region.
Let $\boldsymbol{n}$ be the exterior normal to Γ and let $\phi \in H$ be a given function. However, the function $F(z)$ is in general a multi-valued function.

Without lost of generality, we consider solving Laplace equation with Dirichlet condition on $\Gamma_1,\ldots,\Gamma_l$ and Neumann condition on $\Gamma_{l+1},\ldots,\Gamma_m$. We shall consider the mixed boundary value problem. Define a real function u such that

$$\Delta u = 0 \ a \text{ in } G,$$
$$u = \phi_j \text{ on } \Gamma_j, j = 1,\ldots,l,$$
$$\frac{\partial u}{\partial \boldsymbol{n}} = \phi_j \text{ on } \Gamma_j, j = l+1,\ldots.,m$$

The unique solution $u(z)$ of the mixed boundary value problem can be regarded as a real part of an analytic function $F(z)=u(z)+i\,v(z)$. We define a complex-valued [8,9] function $\hat{A}(s)$ and a real-valued function $\gamma(s)$ for $s \in J_j, j=1,\ldots,l$, by

$$\hat{A}_j(s) = 1, \quad \gamma_j(s) = \phi_j(s),$$

and using the Cauchy-Riemann equations for $s \in J_j$, $j = l+1,\ldots,m$, by

$$\hat{A}_j(s) = -i, \qquad \gamma_j(s) = \int_0^s \phi_j(t)\left|\dot{\eta}_j(t)\right|dt.$$

Let also $\hat{h}(s)$ be the piecewise constant function

$$\hat{h}(s) = \begin{cases} 0, & s \in J_j = [0,2\pi], \quad j = 1,\ldots,l \\ c_j, & s \in J_j = [0,2\pi], \quad j = l+1,\ldots,m \end{cases}$$

where c_j, $j = l+1,\ldots,m$, are undetermined real constants. Thus the boundary values of the function $F(z)$ satisfy the boundary condition [2-4]

$$\text{Re}\left[\hat{A}(s)F(s)\right] = \gamma(s) + \hat{h}(s), \qquad s \in J.$$

The function $F(z)$ can be written as

$$F(z) = \hat{F}(z) - \sum_{j=1}^{m} a_j \log(z - z_j),$$

where $\hat{F}(z)$ is a single-valued analytic function in G, z_j is a fixed point in G and a_j is undetermined real constant, $j=1,2,\ldots,m$ [4,10]. We assume that the function $\hat{F}(z)$ satisfies $\Im \hat{F}(\infty)=0$ in G.

In this paper we shall consider only the case for which $a_j = 0,\ j=1,2,\ldots,m$

Thus the function $\hat{F}(z)$ is a solution of the Riemann-Hilbert problem

$$\text{Re}\left[\hat{A}(s)\hat{F}(s)\right] = \gamma(s) + \hat{h}(s), \qquad s \in J. \tag{10}$$

Let $\hat{F}(\infty) = \hat{c}$ (real constant), then the function

$$g(z) = \hat{F}(z) - \hat{c}$$

is analytic single-valued function in *G.* Thus

$$\hat{A}(s)\hat{F}\big(\eta(s)\big) = \hat{A}(s)g\big(\eta(s)\big) + \hat{A}(s)\hat{c}.$$

Hence (10) can be written as

$$\text{Re}[A(s)g(\eta(s))] = \gamma(s) + h(s),$$

where

$$A(s) = \hat{A}(s),$$
$$h(s) = \hat{h}(s) - Re[\hat{A}(s)\hat{c}].$$

According to Theorem 1, let $\mu = \operatorname{Im} Ag$ (unknown function), then

$$A(s)g(s) = \gamma(s) + h(s) + i\mu(s)$$

where μ is the unique solution of the integral equation

$$\mu - N\mu = -M\gamma, \tag{11}$$

where N and M are defined as in(8) and (9). By obtaining μ, we can get h from

$$h = [M\mu - (I-N)\gamma]/2.$$

(see Theorem 2 of [8]. For more details, see [2-4,7].

5. NUMERICAL IMPLEMENTATIONS

Since the functions A_j and η_j are 2π-periodic, the integrals in the operators $\boldsymbol{N}$ and $\boldsymbol{M}$ in the integral equation (11) can be best discretized on an equidistant grid by the trapezoidal rule [5]. The computational details are similar to previous works [6].

By using the trapezoidal rule with n (an even positive integer) equidistant collocation points on each boundary component, solving the integral equations (11) reduces to solving mn by mn linear systems. Since the integral equations (11) are uniquely solvable, then for sufficiently large values of n the obtained linear systems are also uniquely solvable[1].

In this paper, the linear systems are solved using the Gauss elimination method. By solving the linear systems, we obtain approximations to μ. Hence, we obtain approximations to $\tilde{h}$. Then, we obtain approximation to the constants c, and c_j for $j= l+1,...,m$. Hence, we obtain approximations to the boundary values of the function $g(z)$ from $Ag = \gamma + h + i\mu$. Then the values of $g(z)$ for $z \in G$ will be calculated by the Cauchy integral formula. For points$z$ which are not close to the boundaryΓ, the integrals in the Cauchy integral formula are approximated by the trapezoidal rule. However, for points z near the boundary Γ, the integrand is nearly singular. For the latter case, the integral in the Cauchy integral formula can be calculated accurately using the method suggested.

6. NUMERICAL EXAMPLES

To illustrate this approach, we consider three test regions. By $|u(z)-u_n(z)|$where $u_n(z)$ is the numerical approximation of $u(z)$.The result can be shown in Table1,Table2,Table3.

6.1 Example 1

In this example we consider a unbounded multiply connected region of connectivity 3 unbounded by the three circles

$$\Gamma_1 : \eta_1 = 1 + 0.5e^{-it},$$
$$\Gamma_2 : \eta_2 = 1 - 0.5e^{-it},$$
$$\Gamma_3 : \eta_3 = 0.2e^{-it}$$

We assume that the condition on the boundaries Γ_1, Γ_2 is the Neumann condition and the condition on the boundaries Γ_3 is the Dirichlet condition. In this example we used the exact solution

$$\Re(F) = \Re\left(\frac{1}{z}\right).$$

To obtain $\phi_{1,2}$for the Neumann condition we differentiate with respect to the normal the real part of the exact solution.

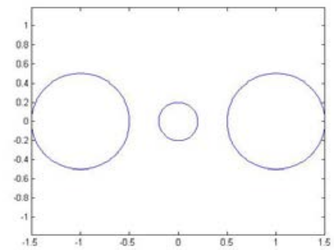

Fig. 2 The unbounded multiply regions of 3-circles connectivities

Table 1 The error norm$|u(z)-u_n(z)|$

n	*Z=0.3*	*Z=0.9i*	*Z=-0.7*	*Z=-0.3i*
16	*1(-03)*	*5(-7)*	*2(-6)*	*2(-7)*
32	*5(-10)*	*8(-19)*	*5(-09)*	*3(-16)*
64	*2(-16)*	*9(-19)*	*3(-17)*	*5(-21)*

6.2 Example

In this example we consider a unbounded multiply connected region of connectivity 4 unbounded by the forth circles

$$\Gamma_1 : \eta_1 = e^{-it}$$
$$\Gamma_2 : \eta_2 = 5 + e^{-it}$$
$$\Gamma_3 : \eta_3 = -5 + e^{=it},$$

We assume that the condition on the boundaries Γ_1, Γ_2 is the Neumann condition and the condition on the boundaries Γ_3 is the Dirichlet condition.In this example we used the exact solution

$$\Re(F) = \Re\left(\frac{1}{z}\right).$$

To obtain $\phi_{1,2}$for the Neumann condition we integrand the real part of the exact solution.

Table 2 The error norm $|u(z)-u_n(z)|$

n	*Z=3*	*Z=3i*	*Z=-3*	*Z=-3i*
16	*1(-03)*	*4(-6)*	*2(-5)*	*4(-7)*
32	*1.4(-6)*	*2(-8)*	*2(-7)*	*2{-16)*
64	*1(-17)*	*1(-17)*	*8(-18)*	*8(-19)*

6.3 Example 3

In this example we consider a unbounded multiply connected region of connectivity 4 unbounded by the fourth circles

$$\Gamma_1 : \eta_1 = 3 + 2i + e^{-it}$$
$$\Gamma_2 : \eta_2 = -3 + 2i + e^{-it}$$
$$\Gamma_3 : \eta_1 = 3 + 2i + e^{-it}$$
$$\Gamma_4 : \eta_4 = 3 - 2i + e^{-it}$$

We assume that the condition on the boundaries Γ_1, Γ_2 is the Neumann condition and the condition on the boundaries Γ_3, Γ_4 is the Dirichlet condition.In this example we used the exact solution

$$\Re(F) = \Re\left(\frac{1}{z+3+2i}\right).$$

To obtain $\phi_{1,2}$for the Neumann condition we differentiate with respect to the normal the real part of the exact solution.

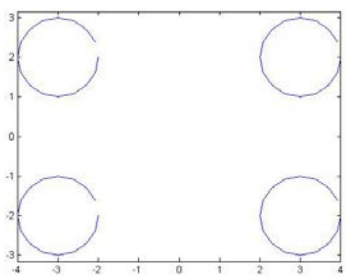

Fig. 3 The unbounded multiply regions of 4 circles connectivities

Table 3 The error norm $|u(z)-u_n(z)|$

n	*Z=-3*	*Z=1+i*	*Z=-0.3*	*Z=0.3i*
16	*4(-6)*	*3(-5)*	*3.2(-5)*	*3(-05)*
32	*1.5(-10)*	*1(-10)*	*1.7(-10)*	*1(-11)*
64	*3.8(-16)*	*1(-16)*	*1.3(-16)*	*8(-17)*

7. CONCLUSION

The uniquely solvable integral equation is derived in this work for the mixed boundary value problem with Dirichlet-Neumann condition on unbounded multiply connected region. The derived boundary integral equation is uniquely solvable and yields directly the boundary value of the solution of the mixed problem.

Mixed boundary value problem is solved numerically on unbounded multiply connected region using the proposed method. The numerical examples illustrate that the proposed method yields approximations of high accuracy.

ACKNOWLEDGEMENT

The authors acknowledge the financial support for this research by the Malaysian Ministry of Higher Education (MOHE) through UTM GUP Vote Q.J130000.7126.01H75 and Ibnu Sina Institute, Universiti Teknologi Malaysia, Johor for facilities.

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