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Partial Sums For Class Of Analytic Functions Defined By Integral Operator

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ABSTRACT

In the present paper, we study the class of analytic functions involving generalized integral operator, which is defined by means of a general Hurwitz Lerch Zeta function denoted by $\Im_{s,b}^{\alpha} f(z)$ with negative coefficients. The aim of the paper is to obtain the coefficient estimates and also partial sums of its sequence $\Im_{s,b}^{\alpha} f(z)$

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| Univalent functions | uniformly starlike functions | Hadamard product | partial sums | fractional derivatives and fractional integrals |

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1. INTRODUCTION

Let A denote the class of all analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ given by the normalized power series of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in U). \quad (1)$$

1.1 Definition: (Owa and. Srivastava [3]) Let the function f be analytic in a simply connected domain of the z-plane containing the origin. The fractional derivative of f of order α is defined by

$$D_z^{\alpha} f(z) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\alpha}} \, dt, \, (0 \le \alpha < 1),$$

where the multiplicity of $(z - t)^{-\alpha}$ is removed by

requiring $\log(z - t)$ to be real when $z - t > 0$.

Using Definition 1.1 and its known extensions involving fractional derivatives and fractional integrals, Owa and Srivastava [3] introduced the operator $\Omega^{\alpha} : A \to A$ as follows:

$$\Omega^{\alpha} f(z) = \Gamma(2-\alpha) z^{\alpha} D_z^{\alpha} f(z), \, (\alpha \ne 2,3,4,...)$$

$$= z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} a_n z^n, (z \in U).$$

1.2 Definition: (Srivastava and Choi [2]) A general Hurwitz Lerch Zeta function $\Phi(z,s,b)$ defined by

$$\Phi(z,s,b) = \sum_{n=0}^{\infty} \frac{z^n}{(n+b)^s},$$

where ($s \in \mathbb{C}, b \in \mathbb{C} - \mathbb{Z}_0^-$ when ($|z| < 1$),

$\Re(b) > 1$ when ($|z| = 1$).

We can write this function as :

$$\Phi^*(z,s,b) = (b^s z \, \Phi(z,s,b)) * f(z),$$

then

$$\Phi^*(z,s,b) = z + \sum_{n=2}^{\infty} \frac{b^s a_n}{(n+b-1)^s} z^n.$$

By using Definitions (1.1) and (1.2), the authors [1] introduced the generalized integral operator $\Im_{s,b}^{\alpha} : A \to A$ as the following:

For $0 \le \alpha < 1$ and $s \in \mathbb{C}, b \in \mathbb{C} - \mathbb{Z}_0^-$

$$\Im_{s,b}^{\alpha} f(z) = (b)^s \Gamma(2-\alpha) z^{\alpha} D_z^{\alpha} \Phi^*(z,s,b), \quad (\alpha \ne 2,3,4,...)$$

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$$= z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} \left(\frac{b}{(n+b-1)}\right)^{s} a_n z^n, (z \in U).$$

Note that :

$$\Im_{0,b}^{0} f(z) \equiv f(z).$$

Special cases of this operator includes:

$\Im_{0,b}^{\alpha} f(z) \equiv \Omega^{\alpha} f(z)$ is Owa and Srivastava operator [3].

$\Im_{s,b+1}^{0} f(z) \equiv J_{s,b} f(z)$, is Srivastav and A. Attiya integral operator[5].

$\Im_{1,1}^{0} f(z) \equiv A(f)(z)$, is Alexander integral operators [6].

$\Im_{s+1,1}^{0} f(z) \equiv L(f)(z)$, is Libera integral operators [7].

$\Im_{1,\delta}^{0} f(z) \equiv L_{\delta}(f)(z)$; $\delta > -1$, is Bernardi integral operator [8].

$\Im_{s,\sigma}^{0} f(z) \equiv I^{\sigma} f(z)$; $(\sigma > 0)$ is Jung- Kim-Srivastava integral operator [9].

$\Im_{-n,1}^{0} f(z) \equiv S^{n}$; $(n \in N)$ is the Salagean derivative operator [13].

$\Im_{-s,b+1}^{0} f(z)$ $(s \in Z)$ multiplier transformations studied by Flett [14].

Finally, for different choices of s, b and α, several operators investigated earlier by other authors Cho and Kim [15], and Lin and Owa [16] are obtained.

By using our integral operator we introduce the following class of A.

A function $f \in A$ is said to be in the class denoted by $SP_{s,b}^{\alpha}(\beta,\gamma), (-1 \le \gamma < 1), \beta \ge 0$ and satisfy

$$\Re\left\{\frac{z(\Im_{s,b}^{\alpha} f(z))'}{(\Im_{s,b}^{\alpha}) f(z)} - \gamma\right\} \ge \beta \left|\frac{z(\Im_{s,b}^{\alpha} f(z))'}{(\Im_{s,b}^{\alpha}) f(z)} - 1\right|,$$

where $s \in □, b \in □ - Z_0$.

By suitably specializing the values of α, β, γ and s the class $SP_{s,b}^{\alpha}(\beta,\gamma)$ reduces to the classes introduced and studied by various authors for example: For $SP_{0,b}^{0}(\beta,\gamma) \equiv SP(\beta,\gamma)$ was introduced Rφnning [10], [2].

2. RESULTS & DISCUSSION

2.1 Coefficient Estimates:

Before stating and proving our main results, we derive a sufficient condition giving the coefficient estimates for the function f to belong to the class $SP_{s,b}^{\alpha}(\beta,\gamma)$ .The result is contained in the following:

2.2 Theorem

A sufficient condition for a function f of the form (1) to be in $SP_{s,b}^{\alpha}(\beta,\gamma)$ is that

$$\sum_{n=2}^{\infty} \left[(1+\beta)n - (\gamma+\beta)\right] \left(\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\right) \left|\left(\frac{b}{n-1+b}\right)^{s}\right| |a_n| \le 1-\gamma,$$

where $s \in □, b \in □ - □_0$ and $(-1 \le \gamma < 1), \beta \ge 0$.

Proof:

It suffices to show that

$$\Re\left\{\frac{z(\Im_{s,b}^{\alpha} f(z))'}{(\Im_{s,b}^{\alpha}) f(z)} - \gamma\right\} - \beta \left|\frac{z(\Im_{s,b}^{\alpha} f(z))'}{(\Im_{s,b}^{\alpha}) f(z)} - 1\right|$$

$$\le (1+\beta) \left|\frac{z(\Im_{s,b}^{\alpha} f(z))'}{(\Im_{s,b}^{\alpha}) f(z)} - 1\right|$$

$$\le \frac{(1+\beta)\sum_{n=2}^{\infty}(n-1)\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\left|\left(\frac{b}{(n+b-1)}\right)^{s}\right| |a_n|}{1-\sum_{n=2}^{\infty}(n-1)\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\left|\left(\frac{b}{(n+b-1)}\right)^{s}\right| |a_n|},$$

$$\sum_{n=2}^{\infty} \left[(1+\beta)n - (\gamma+\beta)\right] \left(\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\right) \left|\left(\frac{b}{n-1+b}\right)^{s}\right| |a_n| \le 1-\gamma,$$

the proof is complete.

2.3 partial sums

In this section we will examine the ratio of a function of the form (1) to its sequence of partial sums defined by $f_1(z) = z$ and $f_k(z) = z + \sum_{n=2}^{k} a_n z^n$ when the coefficients of f are sufficiently small to satisfy the condition (2). We will determine sharp lower bounds for $\Re\left\{\frac{f(z)}{f_k(z)}\right\}, \Re\left\{\frac{f_k(z)}{f(z)}\right\}, \Re\left\{\frac{f'(z)}{f_k'(z)}\right\}$ *and* $\Re\left\{\frac{f_k'(z)}{f'(z)}\right\}$.

2.4 Theorem

Let f be given by (1) satisfying (2), then

$$\Re\left\{\frac{f(z)}{f_k(z)}\right\}\geq 1-\frac{1-\gamma}{\left((1+\beta)(k+1)-(\gamma+\beta)\right)\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^s\right|},$$

$(z\in U).$ (3)

$$\Re\left\{\frac{f_k(z)}{f(z)}\right\}\geq \frac{[(1+\beta)(k+1)-(\gamma+\beta)]\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^s\right|}{[(1-\gamma)+[(1+\beta)(k+1)-(\gamma+\beta)]\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^s\right|},$$

$(z\in U).$ (4)

The results are sharp for every k with the function given by

$$f(z)=z+\frac{(1-\gamma)z^{k+1}}{\left[(1+\beta)(k+1)-(\gamma+\beta)\right]\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^s\right|},$$

$(z\in U, k\geq 0).$ (5)

Proof:

We prove (3).Let f be given by (1) satisfying (2), by sitting

$$\frac{[(1+\beta)(k+1)-(\gamma+\beta)]\left|\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\right|\left|\left(\frac{b}{k+b}\right)^s\right|}{1-\gamma}\left(\frac{f(z)}{f_k(z)}-1-\frac{1-\gamma}{\left((1+\beta)(k+1)-(\gamma+\beta)\right)\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^s\right|}\right)$$

$$=\frac{1+w(z)}{1-w(z)}.$$

We find that

$$w(z)=\frac{\sum_{n=k+1}^{\infty}\frac{[(1+\beta)(k+1)-(\gamma+\beta)]\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^s\right|z^{n-1}_n}{1-\gamma}}{2+2\sum_{n=2}^{k}a_nz^{n-1}+\sum_{n=k+1}^{\infty}\frac{[(1+\beta)(k+1)-(\gamma+\beta)]\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^s\right|a_nz^{n-1}}{1-\gamma}},$$

$$|w(z)|=\frac{\sum_{n=k+1}^{\infty}\frac{[(1+\beta)(k+1)-(\gamma+\beta)]\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^s\right||a_n|}{1-\gamma}}{2-2\sum_{n=2}^{k}|a_n|+\sum_{n=k+1}^{\infty}\frac{[(1+\beta)(k+1)-(\gamma+\beta)]\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^s\right||a_n|}{1-\gamma}}.$$

Now $|w(z)|\leq 1$ if and only if

$$\frac{2((1+\beta)(k+1)-(\gamma+\beta)]\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^s\right|}{1-\gamma}\sum_{n=k+1}^{\infty}|a_n|$$

$$\leq 2-2\sum_{n=2}^{k}|a_n|$$

which is equivalent to

$$\frac{[(1+\beta)(k+1)-(\gamma+\beta)]\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^s\right|}{1-\gamma}\sum_{n=k+1}^{\infty}|a_n|+\sum_{n=2}^{k}|a_n|\leq 1.\ (6)$$

This will hold if we show that left-hand side of (6) is bounded above by

$$\sum_{n=2}^{\infty}\frac{[(1+\beta)n-(\gamma+\beta)]\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\left|\left(\frac{b}{n+b-1}\right)^s\right||a_n|}{1-\gamma}.$$

This is equivalent to showing that

$$\sum_{n=2}^{\infty}\frac{[(1+\beta)n-(\gamma+\beta)]\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\left|\left(\frac{b}{n+b-1}\right)^s\right|-1-\gamma}{1-\gamma}|a_n|+$$

$$\frac{\sum_{n=k+1}^{\infty}(1+\beta)(n-1-k)-(\gamma+\beta)\left(\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\left|\left(\frac{b}{n+b-1}\right)^s\right|-\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^s\right|\right)}{1-\gamma}$$

$|a_n|\geq 0$, (7)

To see that

$$f(z)=z+\frac{(1-\gamma)z^{k+1}}{\left((1+\beta)(k+1)-(\gamma+\beta)\right)\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^s\right|},$$

$(z\in U, k\geq 0),$

gives sharp result, we observe that for $z=re^{\frac{\pi i}{n}}$ that

$$\operatorname{Re}\left\{\frac{f(z)}{f_k(z)}\right\} \geq$$

$$1+\frac{1-\gamma}{[(1+\beta)(k+1)-(\gamma+\beta)]\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^s\right|}z^k,$$

$$\rightarrow$$

$$1-\frac{1-\gamma}{[(1+\beta)(k+1)-(\gamma+\beta)]\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\left|\left(\frac{b}{n+b-1}\right)^s\right|},$$

when $z \rightarrow 1^-$.

To prove the second part of this theorem, we write

$$\frac{(1-\gamma)+[(1+\beta)(k+1)-(\gamma+\beta)]\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^s\right|}{1-\gamma}\Bigg(\frac{f_k(z)}{f(z)}-$$

$$\frac{[(1+\beta)(k+1)-(\gamma+\beta)]\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^s\right|}{(1-\gamma)+[(1+\beta)(k+1)-(\gamma+\beta)]\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^s\right|}\Bigg)$$

$$=\frac{1+w(z)}{1-w(z)}.$$

we find that

$$|w(z)| = \frac{\sum_{n=k+1}^{\infty}\frac{(1-\gamma)+[(1+\beta)(k+1)-(\gamma+\beta)]\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^s\right|\,|a_n|}{1-\gamma}}{2-2\sum_{n=2}^{k}|a_n|+\sum_{n=k+1}^{\infty}\frac{(1-\gamma)+[(1+\beta)(k+1)-(\gamma+\beta)]\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^s\right|\,|a_n|}{1-\gamma}}.$$

Now $|w(z)| \leq 1$ if and only if

$$\frac{2\left((1-\gamma)+[(1+\beta)(k+1)-(\gamma+\beta)]\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^s\right|\right)}{1-\gamma}\sum_{n=k+1}^{\infty}|a_n|$$

$$\leq 2-2\sum_{n=2}^{k}|a_n|,$$

which is equivalent to

$$\frac{(1-\gamma)+[(1+\beta)(k+1)-(\gamma+\beta)]\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^s\right|}{1-\gamma}\sum_{n=k+1}^{\infty}|a_n|+$$

$$\sum_{n=2}^{k}|a_n| \leq 1.$$

Making use of (2) to get (7). Finally, equality holds in (4) for the extremal function f given by (5).This completes the proof.

2.5 Theorem

Let f be given by (1) satisfying (2), then

$$\operatorname{Re}\left\{\frac{f'(z)}{f'_k(z)}\right\} \geq 1-$$

$$\frac{(k+1)(1-\gamma)}{[(1+\beta)(k+1)-(\gamma+\beta)]\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^s\right|}, \quad (8)$$

$$\operatorname{Re}\left\{\frac{f'_k(z)}{f'(z)}\right\} \geq$$

$$\frac{\sum_{n=k+1}^{\infty}([(1+\beta)(k+1)-(\gamma+\beta)]\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^s\right|}{(1-\gamma)(k+1)+[(1+\beta)(k+1)-(\gamma+\beta)]\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^s\right|}. \quad (9)$$

The results are sharp for every k with the function give

$$f(z)=z+\frac{(1-\gamma)z^{k+1}}{\left[(1+\beta)(k+1)-(\gamma+\beta)\right]\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^s\right|},$$

$(z \in U, k \geq 0)$.

Proof: To prove the result (8), define thefunction

w(z) by

$$\frac{[(1+\beta)(k+1)-(\gamma+\beta)]\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^s\right|}{(1-\gamma)}\Bigg(\frac{f'_k(z)}{f'(z)}-$$

$$\left(1-\frac{(1-\gamma)(k+1)}{[(1+\beta)(k+1)-(\gamma+\beta)]\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^s\right|}\right)\Bigg),$$

where

$$w(z)=\frac{\sum_{n=k+1}^{\infty}\frac{n[(1+\beta)(k+1)-(\gamma+\beta)]\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^{s}\right|a_n z^{n-1}}{(1-\gamma)(k+1)}}{2+2\sum_{n=2}^{k} n a_n z^{n-1}+\sum_{n=k+1}^{\infty}\frac{n[(1+\beta)(k+1)-(\gamma+\beta)]\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^{s}\right|a_n z^{n-1}}{(1-\gamma)(k+1)}}.$$

Now $|w(z)|\le 1$ if and only if

$$\frac{\left([(1+\beta)(k+1)-(\gamma+\beta)]\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^{s}\right|\right)}{(1-\gamma)(k+1)}\sum_{n=k+1}^{\infty} n|a_n|$$

$$+\sum_{n=2}^{k} n|a_n|\le 1 .$$

Since the LHS of (8) is bounded above by

$$\sum_{n=2}^{\infty}\frac{(1+\beta)n-(\gamma+\beta)]\frac{\Gamma(n+1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)}\left(\frac{b}{n+b-1}\right)^{s}}{1-\gamma}|a_n| .$$

The proof is complete.

To prove the second part of this theorem, we write

$$\frac{(1-\gamma)(k+1)+[(1+\beta)(k+1)-(\gamma+\beta)]\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^{s}\right|}{1-\gamma}\left(\frac{f'_k(z)}{f'(z)}-\right.$$

$$\left.\frac{[(1+\beta)(k+1)-(\gamma+\beta)]\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^{s}\right|}{(1-\gamma)(k+1)+[(1+\beta)(k+1)-(\gamma+\beta)]\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^{s}\right|}\right)=\frac{1+w(z)}{1-w(z)},$$

we find that

$$w(z)=\frac{\sum_{n=k+1}^{\infty}1+\frac{[(1+\beta)(k+1)-(\gamma+\beta)]\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^{s}\right|a_n z^{n-1}}{(1-\gamma)(k+1)}}{2+2\sum_{n=2}^{k} n\, a_n z^{n-1}+\sum_{n=k+1}^{\infty}1+\frac{[(1+\beta)(k+1)-(\gamma+\beta)]\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^{s}\right|a_n z^{n-1}}{(1-\gamma)(k+1)}}.$$

Now $|w(z)|\le 1$ if and only if

$$\frac{2\left(1+[(1+\beta)(k+1)-(\gamma+\beta)]\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^{s}\right|\right)}{(1-\gamma)(k+1)}\sum_{n=k+1}^{\infty} n|a_n|$$

$$\le 2-2\sum_{n=2}^{k} n|a_n| ,$$

which is equivalent to

$$\left(1+\frac{[(1+\beta)(k+1)-(\gamma+\beta)]\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^{s}\right|}{(1-\gamma)(k+1)}\right)\sum_{n=k+1}^{\infty} n|a_n|$$

$$+\sum_{n=2}^{k} n|a_n|\le 1$$

From the condition (2), it is sufficient to show that

$$\sum_{n=k+1}^{\infty}\frac{((1-\gamma)(k+1)+[(1+\beta)(k+1)-(\gamma+\beta)](n-k-1))\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^{s}\right|}{(1-\gamma)(k+1)}|a_n|$$

$$+\sum_{n=2}^{k}\frac{[(1+\beta)(k+1)-(\gamma+\beta)]\frac{\Gamma(k+2)\Gamma(2-\alpha)}{\Gamma(k+2-\alpha)}\left|\left(\frac{b}{k+b}\right)^{s}\right|}{(1-\gamma)(k+1)}|a_n|\ge 0 .$$

which gives (9). The bound in (9) is sharp for all k ∈ N with the extremal function (5). Special cases of Theorems 2.4, 2.5 by setting $\alpha=0$, $s=0$ and $\beta=0$ can be found in [11].

2.5 Corollary

Let f be given by (1) satisfying

$$\sum_{n=2}^{\infty}|(n-\gamma)|\le 1-\gamma,$$

Then

$$\operatorname{Re}\left\{\frac{f(z)}{f_k(z)}\right\}\ge\frac{k}{k+1-\gamma}(z\in U),$$

$$\Re\left\{\frac{f_k(z)}{f(z)}\right\}\ge\frac{k+1-\gamma}{k+2-2\gamma}(z\in U),$$

$$\Re\left\{\frac{f'(z)}{f'_k(z)}\right\} \geq \frac{k\gamma}{k+1-\gamma} (z \in U),$$

and

$$\Re\left\{\frac{f'(z)}{f'_k(z)}\right\} \geq \frac{k+1-\gamma}{(k+1-\gamma)+(k+1)(1-\gamma)} (z \in U).$$

The results are sharp for every k with the function given by

$$f(z) = z + \frac{1-\gamma}{k+1-\gamma} (z \in U) \quad .$$

3. CONCLUSION

Our results certainly generalized several results obtained earlier. Therefore, partial sums of functions in the unit disk defined by a general integral operator are obtained The operator defined can be extended and can solve many new results and properties.

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