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Review on Fuzzy Difference Equation

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ABSTRACT

Fuzzy difference equation has been introduced by Kandel and Byatt in 1978. This topic has been growing rapidly for many years. Fuzzy difference forms is suitable for uncertainty or vagueness problems such as mathematical modelling, finance or else and it also applied in engineering, economics, science and etc. In this paper, we review the application of fuzzy difference equations that has been used before. We also give some ideas to relate from this type-1 fuzzy difference equation to type-2 fuzzy. The using of type-2 fuzzy is for more uncertainties and it is really suitable for problems in finance.

| Fuzzy difference equation | Type-2 fuzzy | Finance |

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1. INTRODUCTION

Fuzzy sets theory is a tool and used for a problems that have uncertainty or vagueness. Fuzzy sets theory was originally introduced by Lotfi A. Zadeh in year 1965 [1] that led to definition fuzzy number and its implementation in fuzzy control [2].

A fuzzy difference equation is an equation that contains sequence differences. To solve the difference equation is by finding a sequence that satisfies the equation. The sequence that satisfies the equation is called sequence a solution of the equation.

A fuzzy difference equation is a difference equation where constants and the initial values are fuzzy numbers and its solutions are sequences of fuzzy numbers. Fuzzy difference equations have been growing rapidly developed for the many years which are appears naturally as discrete analogous and as numerical solutions of differential equations.

The fuzzy difference equations initially introduced by Kandel and Byatt [3,4]. Also for the fuzzy difference equations and initial value problem (Cauchy problem) were rigorously explained by Kaleva [5,6]. Zhang, Yang, and Liao [7] on their study the existence of positive solution in fuzzy difference equation. They proof that the positive solutions are bounded and persists.

They used fuzzy numbers for the interest rate, the taxation, and the inflation due to the fluidity and the uncertainty existing in financial market. They comparing their method with an existing method presented by Buckley [9].

Fuzzy difference equations also suitable in finance problem. Chrysafis, Papadopoulos, Papaschinopoulos [8] in their study about the fuzzy difference equation of finance.Their research is in finance which is about the alternative methodology to study the time value of money, the method of fuzzy difference equation.

In this paper, we will review the fuzzy difference equations and their application in any field of problems. For the section 2, we define the theory of classical sets, fuzzy sets, and fuzzy difference equations theory. Also, we give the definition of type-2 fuzzy sets theory.

For the third section, we will review about the fuzzy difference equations which are the form of fuzzy difference equations and their applications that have been used before.

Then, we will give some ideas of using type-2 fuzzy sets and relate type-1 fuzzy difference equation to type-2 fuzzy difference equation. Type-2 fuzzy is used for more uncertainty or more vagueness. Type-2 fuzzy sets are the extension of type-1 fuzzy sets with an additional dimension represents the uncertainty about the degree of membership. The concept of type-2 fuzzy sets was introduced by Zadeh [10]. The using of type-2 fuzzy is suitable in financial problem.

2. PRELIMINARIES

2.1 Classical Sets

Definition 1: Suppose X and Y are two different universes of discourse (information), if x is contained in X and corresponds to y in Y, it is 'termed mapping' from X to Y or $f: X \to Y$. As mapping, the characteristics (indicator) function $\mu_A(x)$ is defined as:

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$$\mu_A(x) = \begin{cases} 1, & x \in X \\ 0, & otherwise \end{cases}$$

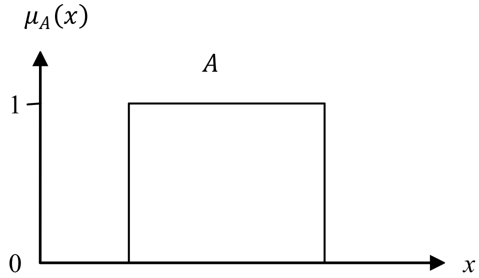

Fig. 1 Membership function

where $\mu_A(x)$ express membership in set A for the element x in the universe. This is a mapping from an element in x in the universe X to one of the two elements in universe Y, i.e. to the elements 0 or 1 as shown in Figure 1. For any set A defined on X, there exists a function-theoretic set, called a value set, denoted by $V(A)$ under the mapping of the characteristic function $\mu_A(x)$ By convention, null set $\emptyset$ is assigned a value 0 and the whole set X assigned a value 1.

2.2 Fuzzy Sets

Fuzzy set is sets whose elements have membership function. Fuzzy sets were introduced by Lotfi A. Zadeh in 1965 [1]. Fuzzy sets also as an extension of the classical set.

Definition 2: Let fuzzy set A in X is defined as follow

$$\tilde{A} = \{(x, \mu_A(x)) | x \in X\}$$

Fuzzy set defined that membership set as a p robability distribution. General rule for fuzzy set can be state as:

$$f: [0,1]^n \rightarrow [0,1]$$

which n is number of probability.

Definition 3: [8] A is fuzzy number if $A: R \rightarrow [0,1]$ statisfies the following condition:

i. A is normal
ii. A is a convex fuzzy set
iii. A is upper semicontinuous,
iv. Support of A, $\cup_{a\in[0,1]} [A]_a = \{x: A(x) > 0\}$ is compact.

Then the a-cuts of A are closed intervals. We known that $[A]_a = B_a \forall a \in [0,1]$ for arbitrary fuzzy sets A and B then $A = B$

Definition 4: [11] A sequence $X = (X_n)$ of fuzzy numbers is a function X from the set N of all positive integers into $L(R)$. The fuzzy number X_n denotes the value of the function at $n \in N$ and is called the nth term of the sequence.

Definition 5: [11] A sequence $\Delta X = (\Delta X_n)$ of a fuzzy numbers is said to be convergent to the fuzzy number X_0, written as $\lim_n \Delta X_n = X_0$, if for every $\varepsilon > 0$ there exists a positive integer N such that

$$\bar{d}(\Delta X_n, X_n) < \varepsilon \text{ for } \; n > N$$

Let $c(\Delta)$ denote the set of all convergent difference sequences of fuzzy numbers.

Definition 6: [11] A sequence $\Delta X = (\Delta X_n)$ of fuzzy numbers is said to be bounded if the set $(\Delta X_n : n \in N)$ of fuzzy numbers is bounded, where $\Delta X = (X_n - X_{n+1})$. Let $m(\Delta)$ denote the set of all bounded difference sequences of fuzzy numbers.

Definition 7: [15] Let x_n be a sequence of positive fuzzy numbers such that

$$[x_n]_a = [L_{n,a}, R_{n,a}], \quad a \in (0,1], \quad n = 0,1, \dots,$$

and let x be a positive fuzzy number such that

$$[x]_a = [L_a, R_a], \quad a \in (0,1].$$

We say that x_n nearly converges to x with respect to D as $n \rightarrow \infty$ if for every $\delta > 0$, there exists a measurable set T, $T \subset (0,1]$, of the measure less than δ such that

$$\lim D_T(x_n, x) = 0, \quad as \; n \rightarrow \infty,$$

Where

$$D_T(x_n, x) = \sup_{a\in(0,1]-T} \left\{ \max\{ |L_{n,a}, L_a|, |R_{n,a}, R_a| \} \right\}$$

If $T = \emptyset$, we say that x_n converges to x with respect to D as $n \rightarrow \infty$.

2.3 Difference Equation

Definition 8: Given constant α and β, a difference equation of the form

$$x_{n+1} = \alpha x_n + \beta,$$

$n = 0,1,2, \dots$ is called a first-order linear equation difference equation. A procedure analogous to the method we used to solve $x_{n+1} = \alpha x_n$ will enable to solve this equation as well. Namely,

$$\begin{aligned} x_n &= \alpha x_{n-1} + \beta \\ &= \alpha(\alpha x_{n-2} + \beta) + \beta \end{aligned}$$

$$= \alpha^2 x_{n-2} + \beta(\alpha + 1)$$
$$= \alpha^2(\alpha x_{n-3} + \beta) + (\alpha + 1)$$
$$= \alpha^3 x_{n-3} + \beta(\alpha^2 + \alpha + 1)$$
$$\vdots$$
$$= \alpha^n x_0 + \beta(\alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha^2 + \alpha + 1).$$

Note that $\alpha = 1$, this gives

$$x_n = x_0 + n\beta,$$

$n = 0,1,2,\ldots$ as the solution of the difference equation $x_{n+1} = x_n + \beta$. is $\alpha \neq 1$, known that

$$\alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha^2 + \alpha + 1 = \frac{1-\alpha^n}{1-\alpha}.$$

Hence

$$x_n = \alpha^n x_0 + \beta\left(\frac{1-\alpha^n}{1-\alpha}\right),$$

$n = 0,1,2,\ldots$ is the solution of the first-order linear difference equation $x_{n+1} = \alpha x_n + \beta$ when $\alpha \neq 1$.

Definition 9: [16] The equation

$$u_{n+k} = f(u_{n+k-1}, u_{n+k-2}, \ldots, u_n),$$

For a given function f and unknown quantities u_i, $i = 0,1,\ldots$ is called a difference equation of order k.

When the equation is of the form:

$$a_k u_{n+k} + a_{k-1} u_{n+k-1} + \cdots + a_0 u_n = b, \quad a_0 u_k \neq 0$$

it is called a linear difference equation.

According to whether the coefficients and the right hand side of the equation depend on n or not, it is called an equation with variable or constant coefficients respectively.

When the right hand side $b \neq 0$, then the equation $u_{n+k} = f(u_{n+k-1}, u_{n+k-2}, \ldots, u_n)$ is non-homogeneous, while for $b = 0$, i.e

$$a_k u_{n+k} + a_{k-1} u_{n+k-1} + \cdots + a_0 u_n = 0$$

the equation is called a linear homogeneous difference equation.

2.4 Fuzzy Difference Equation

In G. Papaschinopoulos and B. K. Papadopoulos [12] studied the following fuzzy difference equation:

Definition 10:

$$x_{n+1} = A + \frac{B}{x_n}$$

where (x_n) is sequences of fuzzy numbers and A, B, x_0 are positive fuzzy numbers.

Zhang *et. al.* [7] on their studied of fuzzy nonlinear difference equation is

Definition 11:

$$x_{n+1} = A + \sum_{i=0}^{k} \frac{B_i}{x_{n-i}}, \quad n = 0,1,\cdots,$$

Where (x_n) is a sequence of positive fuzzy number, A, B_i, and the initial values $x_{-k}, x_{-k+1}, \ldots, x_0$ are positive fuzzy numbers, $k \in \{0,1,2,\ldots\}$.

Chrysafis *et. al* [8] used fuzzy difference equation to solved the elementary compound interest in financial problems.

Definition 12: Consider F_n the fuzzy amount of money on time period n and I the fuzzy interest rate. Thus the explicit solution for this linear difference equation is

$$F_n = F_0(1 + I)^n$$

Definition 13: [13] Consider the fuzzy differential equation with piecewise constant argument of the form:

$$x' = p_x(t) + qx([t]), \quad t \in [0, \infty),$$

Where p, q are constant real number, the initial value $x(0) = c_0$ is a fuzzy number such that $[x(0)]_a = [c_0]_a [L_a, R_a]$, $a \in (0,1]$, and $[t]$ is the greater integer function.

Let $x: [0, \infty) \to E$, be a fuzzy map where E is the set of fuzzy numbers such that $[x(t)]_a = [L_a(t), R_a(t)]$, $a \in (0,1]$ holds, so that $x(t)$ is a solution of $x' = px(t) + qt([t])$, $t \in [0, \infty)$, if the following conditions are satisfied:

i. For each $a \in (0,1], L_a(t), R_a(t)$ are continuous functions on $[0,\infty)$.
ii. The derivatives $L'_a(t), R'_a(t)$ exist at any $t \in [0, \infty)$, may except at $t = n$, $n = 0,1,2,\ldots$, where one-sided derivatives exist.
iii. For each $a \in (0,1], (L'_a(t), R'_a(t))$ defines a fuzzy number, such that: $[x'(t)]_a = [L'_a(t), R'_a(t)]$, $a \in (0,1)$, $t \in [0, \infty)$,
iv. $x(t)$ satisfies $x' = px(t) + qx([t])$, $t \in [0, \infty)$ at every interval $[n, n+1)$, $n = 0,1,\ldots$

2.5 Type-2 Fuzzy Sets

Definition 14: [14] A type-2 set, denoted $\tilde{A}$, is characterized by a type-2 membership function $\mu_{\tilde{A}}(x, u)$, where $x \in X$ and $u \in J_x \subseteq [0,1]$,

$$\tilde{A} = \{((x, u), \mu_{\tilde{A}}(x, u)) | \forall x \in X, \forall u \in J_x \subseteq [0,1]\}$$

in which $0 \leq \mu_{\tilde{A}}(x, u) \leq 1$. $\tilde{A}$ can be also expressed as

$$\tilde{A} = \int_{x\in X}\int_{u\in J_x} \mu_{\tilde{A}}(x,u)/(x,u), \quad J_x \subseteq [0,1]$$

Where $\int\int$ denoted union over all admissible x and u. For discrete universes of discourse $\int$ is replaced by $\sum$.

In Definition 14, the first restriction that $\forall u \in J_x \subseteq [0,1]$is consistent with the type-1 constraint that $0 \le \mu_A(x) \le 1$, i.e., when uncertainties disappear s type-2 membership function must reduce to a type-1 membership function, in which case the variable u equals $\mu_A(x)$ and $0 \le \mu_A(x) \le 1$.

The second restriction that $0 \le \mu_{\tilde{A}}(x,u) \le 1$ is consistent with the fact that the amplitudes of a membership function should lie between or be equal to 0 and 1.

Example: [14]

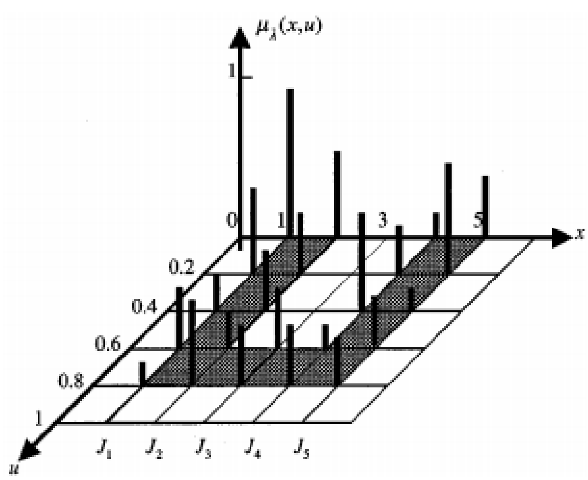

Fig. 2 Example of a t ype-2 membership function. The shaded area is footprint of uncertainty (FOU)

Figure 2 depicts $\mu_{\tilde{A}}(x,u)$ for x and u discrete. In particular, $X = \{1,2,3,4,5\}$ and $U = \{0.2, 0.4, 0.6, 0.8\}$.

Definition 15: [17]Uncertainty in the primary memberships of an interval type-2 fuzzy set consists of a bounded region named as *Footprint of Uncertainty (FOU)*. It is the union of all primary membership, i.e.

$$FOU(A) = \bigcup_{x\in X} J_x$$

This is a vertical-slice representation of *FOU*, because each of primary membership is a vertical slice.

Definition 16: [17] The Upper Membership Function (UMF) and the Lower Membership Function (LMF) of A are two type-1 membership functions that bound the FOU. The UMF is associated with upper bound of $FOU(A)$ and is denoted as $\mu_A(x), \forall x \in X$, and the LMF is associated with the lower bound of $FOU(A)$ and is denoted as $\underline{\mu}_A(x), \forall x \in X$, i.e.

$$\bar{\mu}_A(x) = \overline{FOU(A)} \qquad \forall x \in X$$

$$\underline{\mu}_A(x) = \underline{FOU(A)} \qquad \forall x \in X$$

For an interval type-2 fuzzy set A can denoted as $\sum_{i=1}^{n}\left[\underline{\mu}_A(x_i), \bar{\mu}_A(x_i)\right]/x_i$ (in discrete situation) or $\int_{x\in X}[\underline{\mu}_A(x_i), \bar{\mu}_A(x_i)]\,/x$ (in continuous situation).

3. REVIEW OF FUZZY DIFFERENCE EQUATION AND SOME IDEA TO APPROACH TYPE-2 FUZZY DIFFERENCE EQUATION

3.1 Review of Fuzzy Difference Equation

Deeba and Korvin [11] have presented the model of analysis of CO_2 level in the blood by using a concept of fuzzy difference equation. They consider the model to determine the carbon dioxide (CO_2) level in blood and also consider fuzzy analog of the linearized modes as a method since there is many measurements and factors are imprecise. Their studied also shown the results that classical case was reduced when the fuzzy quantities are replaced by crisp ones.

G. Stefanidou and G. Papaschinopoulos [15] have studied about trichotomy, stability, and oscillation of a fuzzy difference equation. In their paper, they have studied about the trichotomy character, the stability, and the oscillation behaviour of the positive solutions of the fuzzy difference equation.

The development of fuzzy difference equations is does not stop on its own, but its effectiveness and its use continues to be enhanced, especially in finance. Since in the fields of science such as finance are uncertainties, then in 2008 Chrysatis *et. al* [8] have used fuzzy difference equations in the field of finance. They were presented an alternative methodology to study the time value of money, the method of fuzzy difference equations. They use fuzzy numbers for the factors that affect financial market such as interest rate, the taxation, and the inflation also the extra deposits during the life of the accounts. Then, they use the sequences of fuzzy numbers as the solution for the problems. Its shown that applicability of fuzzy difference equations in the field of science such that in finance.

For the following in Zhang, Yang, and Liao [7] studied the existence of positive solution to fuzzy difference equation $x_{n+1} = A + \sum_{i=0}^{k}\frac{B_i}{x_{n-i}},\ n = 0,1,\ldots$. They also prove the solutions are bounded and persists, and also the equation has a unique positive equilibrium which is asymptotically stable.

3.2 Some Idea of Type-2 Fuzzy System

As we know, type-2 fuzzy can manage more uncertainty. For example in the field of finance, as we know that there is many of uncertainty we've found. Chraysafis et. al [8] studied that the balance in account is effected by many factors such as interest rate, inflation, and also the taxation.

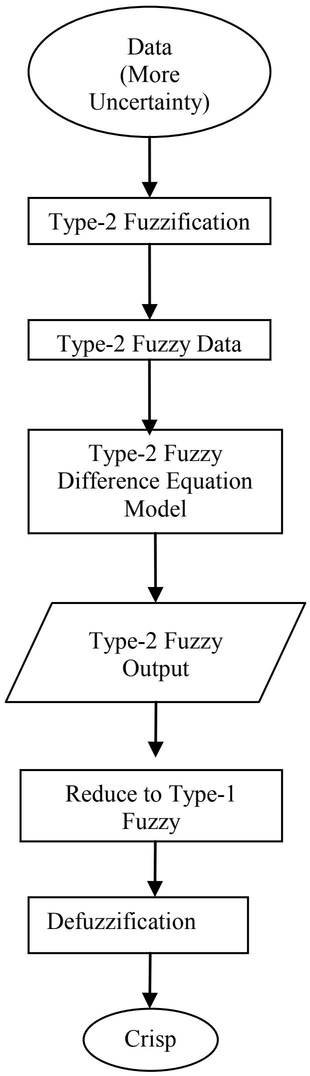

Chart 1 Idea of Type-2 Fuzzy System

If there have more uncertainty such as the factor interest rate, which is in that also have more interest that the customer do not know. So, we can use type-2 fuzzy to reduce the uncertainties. Here, we give some ideas to come up the type-2 fuzzy difference equation:

Idea of type-2 fuzzy system step is as follow:

Step 1: Read data with type-2 fuzzy definitions.
Step 2: Use type-2 fuzzification process.
Step 3: Type-2 fuzzy data form.
Step 4: Use type-2 fuzzy difference equation model.
Step 5: Process type-2 fuzzy output.
Step 6: Reduce to type-1 fuzzy output.
Step 7: Use defuzzification.
Step 8: Get crisp as a solution.

4. CONCLUSION

In this paper we have show the definitions of classical sets, fuzzy sets, difference equation, fuzzy difference equations and type-2 fuzzy. This paper also has reviewed the fuzzy difference equations and the applications that have been used before. Fuzzy difference equation is useful in any fields and most effectively used in the problems which involve the time value.

Furthermore, we give some idea of using type-2 fuzzy difference equation system. As we know, type-2 fuzzy can solve the problems that have more uncertainty. Type-2 fuzzy difference equation is suggested to be use in the field of finance because it can solve problems that have more vagueness and more uncertainty and also the problems that involve in time period.

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