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RESEARCH ARTICLE

# Mathematical analysis of plankton population dynamics

Fatin Nadiah Yussof [a], Normah Maan [a], Nadzri Reba [b,*]

[a] Department of Mathematics, Faculty of Science, Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor, Malaysia
[b] Faculty of Geoinformation and Real Estate, Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor, Malaysia

* Corresponding author: normahmaan@utm.my

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**Abstract**

Harmful algal blooms (HABs) event that causes enormous economic loss and health effect raises concerns among environmentalists. In this paper, a mathematical model of interaction between nutrient, toxin-producing phytoplankton (TPP), non-toxic phytoplankton (NTP), zooplankton, and toxic chemicals is proposed to study on how the process of these HABs occurred. The model of interaction is represented by Ordinary Differential Equations (ODEs) and stability analysis of the model is conducted. Several conditions for the system to be stable around trivial and interior equilibrium point are obtained. From the analysis, it is observed that under nutrient limitation, the amounts of toxic chemicals secreted out by the TPP are increased. As a result, NTP population and zooplankton population are affected by the situation. If this situation is prolonged, this will result in the extinction of both populations. Overall, this study shows that TPP release more toxic chemicals when the nutrient is limited and gives a better understanding on the occurrence of HABs event.

***Keywords***: Stability analysis, harmful algal blooms, toxin-producing phytoplankton, nutrient limitation



## INTRODUCTION

In the marine ecosystem, there are two forms of plankton community which are plant and animal. Phytoplankton is plant forms of plankton community while zooplankton is an animal form of plankton which feeds on phytoplankton. Generally, phytoplankton can be divided into two, either it is toxin-producing phytoplankton (TPP) or non-toxic phytoplankton (NTP). Toxic chemicals released by TPP cause tremendous effect to the marine organism such as shellfish and fish where the toxic will accumulate in the filter feeder and cause food poisoning to the human [3,17]. Toxic chemicals that usually produced by TPP are okadaic acid (OA) and dinophysistoxin (DTX-1) [1]. Over time, the secretion of these toxic chemicals by TPP may result in harmful algal blooms which gives adverse effect to the marine ecosystem and human health.

For microalgae, [2] concluded that the growth of competing algae can be controlled through the toxin production. [5] conducted a field study as well as analyzing mathematical model and claimed that the survival of weak competitors is enhanced by the toxin allelopathy. [4] have shown that TPP population can control the bloom of NTP. Some dinoflagellates-released toxin such as *Alexandrium tamarense* and *Alexandrium fundyense* decrease the production of toxin under deficient nitrogen condition while under phosphorus limited condition the dinoflagellates increase in production of toxin [6-8]. Besides, other studies have shown that dinoflagellates such as *Procentrum lima* and *Dinophysis acuminate* increase the production of toxin when there are limited nitrogen and phosphorus condition [9-11].

It is important to model on plankton communities especially on HAB and co-existence of species since it is lacking in the literature. Mathematical models can give better understanding of the interaction between the nutrient and plankton population. The complexity of an ecosystem response to variant conditions also can be recognized through the developed model with ordinary differential equations. Several mathematical models that described these methods were recently developed [4-5,12,18-21]. In this paper, the developed model is modified from [4] model where the model study the effect of nutrient limitation on toxic production. However, the model does not include and study the interaction of the developed model with zooplankton population. Therefore, our concern is to include the zooplankton population into the model system and investigate the interaction of zooplankton in the system.

The flow of this paper is as follows: Section 2 presents the model; Section 3 shows all the conditions for the existence of the equilibrium points in natural phenomena; Section 4 analyzes the local stability of each equilibrium points and the analysis is implemented using Mathematica software; Section 5 presents the numerical simulations of the developed model; Section 6 discusses on the result obtained; and finally, Section 7 concludes this paper.

## MATHEMATICAL MODEL

This model system is comprised of five ordinary differential equations which are concentration of nutrient levels $N(t)$, TPP populations $P_1(t)$, NTP populations $P_2(t)$, zooplankton populations $P_3(t)$, and toxin concentration $Z(t)$ present at time $t$. Let $A$ be the constant rate of nutrient flows into the system while $d$ is the rate of nutrient loss from the system. Assumptions are made in which all the parameter values are defined as follows:

| | |
|---|---|
| $m_1$ | = maximum nutrient uptake rates for TPP |
| $m_2$ | = maximum nutrient uptake rates for NTP |
| $a$ | = mortality rates of zooplankton due to toxin |
| $a_1$ | = the half saturation constant |
| $a_2$ | = the half saturation constant |
| $r_1$ | = mortality rates of NTP due to toxin |

$r_2$ = mortality rates of zooplankton due to toxin
$k$ = toxic production rate
$k_1$ = conversion factor from TPP to zooplankton
$k_2$ = conversion factor from NTP to zooplankton
$\delta$ = natural death rate of TPP
$\gamma$ = natural death rate of NTP
$\theta$ = toxic washout rate
$\beta_1$ = maximum zooplankton ingestion rate on TPP
$\beta_2$ = maximum zooplankton ingestion rate on NTP
$\varepsilon$ = natural death rate of zooplankton

- Michaelis-Menten functions $\frac{N}{a_1+N}$ and $\frac{N}{a_2+N}$ are used to model the nutrient uptake for growth of TPP and NTP, respectively, in order to illustrate that nutrient concentration level influences the phytoplankton nutrient uptake rate [15]. For growth and nutrient-limited uptake, the equations involved follow the standard representation form [16].
- The functional response of zooplankton that feeds on TPP and NTP is described by the law of predation of Holling type-II [13,14].
- The amount of toxic chemicals released by TPP are higher when the concentration of nutrient becomes limited. Therefore, this phenomenon is described by a monotonic decreasing function of nutrient, $\frac{kP_1}{a+N}$ .
- Toxic chemicals released by TPP gives adverse effect to the existence of NTP and zooplankton population where the toxic chemicals caused the population die.

The model suggested in this work is as follows:

$$\begin{aligned}
\frac{dN}{dt} &= A - dN - \frac{m_1 N P_1}{a_1+N} - \frac{m_2 N P_2}{a_2+N} \\
\frac{dP_1}{dt} &= \frac{m_1 N P_1}{a_1+N} - \delta P_1 - \frac{\beta_1 P_1 P_3}{a_1+P_1} \\
\frac{dP_2}{dt} &= \frac{m_2 N P_2}{a_2+N} - \gamma P_2 - \frac{\beta_2 P_2 P_3}{a_2+P_2} - r_1 P_2 Z \\
\frac{dP_3}{dt} &= \frac{k_1 \beta_1 P_1 P_3}{a_1+P_1} + \frac{k_2 \beta_2 P_2 P_3}{a_2+P_2} - \varepsilon P_3 - r_2 P_3 Z \\
\frac{dZ}{dt} &= \frac{kP_1}{a_1+N} - \theta Z
\end{aligned} \qquad (1)$$

System (1) is subject to the following initial conditions:

$$N(0)>0, P_1(0)>0, P_2(0)>0, P_3(0)>0, Z(0)>0 .$$

## Existence of equilibrium points

The system (1) possesses the following equilibrium points:

- The plankton and toxin free equilibrium $E_1=(N^{(1)},0,0,0,0)$ always exists if

$$N^{(1)} = \frac{A}{d}$$

- The NTP and zooplankton free equilibrium $E_2=(N^{(2)},P_1^{(2)},0,0,Z^{(2)})$ always exists if

$$N^{(2)} = -\frac{a_1\delta}{\delta - m_1}, P_1^{(2)} = \frac{A\delta + a_1 d\delta - Am_1}{\delta(\delta - m_1)},$$

$$Z^{(2)} = \frac{-A\delta k - a_1 d\delta k + Akm_1}{a_1 m_1 \delta\theta}$$

## Local stability analysis of equilibrium points

A system is locally asymptotically stable if all the solutions approaching the equilibrium points as $t \to \infty$. The local stability of the equilibrium points is investigated by finding the eigenvalues of the associated Jacobian matrices at system (1).

The Jacobian matrix at $E_1=(N^{(1)},0,0,0,0)$ is

$$J_1 = \begin{bmatrix}
-d & -\frac{m_1 N^{(1)}}{a_1+N^{(1)}} & -\frac{m_2 N^{(1)}}{a_2+N^{(1)}} & 0 & 0 \\
0 & -\delta + \frac{m_1 N^{(1)}}{a_1+N^{(1)}} & 0 & 0 & 0 \\
0 & 0 & -\gamma + \frac{m_2 N^{(1)}}{a_2+N^{(1)}} & 0 & 0 \\
0 & 0 & 0 & -\varepsilon & 0 \\
0 & \frac{k}{a_1+N^{(1)}} & 0 & 0 & -\theta
\end{bmatrix}$$

Eigenvalues of $J_1$ is given by:

$$\lambda_N = -d, \lambda_{P_1} = -\delta + \frac{m_1 N^{(1)}}{a_1+N^{(1)}}, \lambda_{P_2} = -\gamma + \frac{m_2 N^{(1)}}{a_2+N^{(1)}},$$
$$\lambda_{P_3} = -\varepsilon, \lambda_Z = -\theta$$

The equilibrium point $E_1$ is locally asymptotically stable if the condition along with the following conditions are satisfied:

$$\frac{m_1 N^{(1)}}{a_1+N^{(1)}} < \delta$$
$$\frac{m_2 N^{(1)}}{a_2+N^{(1)}} < \gamma$$

## Biological interpretation

If the respective maximal growth rates $\frac{m_1 N^{(1)}}{a_1+N^{(1)}}$ and $\frac{m_2 N^{(1)}}{a_2+N^{(1)}}$ of TPP and NTP populations are less than their respective corresponding natural removal rates ($\delta$ and $\gamma$) and the amount of nutrient is stabilized at $N^{(1)}$, both TPP and NTP populations will undergo extinction.

The Jacobian matrix at $E_2=(N^{(2)}, P_1^{(2)}, 0, 0, Z^{(2)})$ is

$$J_2 = \begin{bmatrix} -d + \frac{m_1 N^{(2)} P_1^{(2)}}{(a_1 + N^{(2)})^2} - \frac{m_1 P_1^{(2)}}{a_1 + N^{(2)}} & -\frac{m_1 N^{(2)}}{a_1 + N^{(2)}} & -\frac{m_2 N^{(2)}}{a_2 + N^{(2)}} & 0 & 0 \\ -\frac{m_1 N^{(2)} P_1^{(2)}}{(a_1 + N^{(2)})^2} + \frac{m_1 P_1^{(2)}}{a_1 + N^{(2)}} & -\delta + \frac{m_1 N^{(2)}}{a_1 + N^{(2)}} & 0 & -\frac{\beta_1 P_1^{(2)}}{a_1 + P_1^{(2)}} & 0 \\ 0 & 0 & -\gamma + \frac{m_2 N^{(2)}}{a_2 + N^{(2)}} - r_1 Z^{(2)} & 0 & 0 \\ 0 & 0 & 0 & \frac{\beta_1 k_1 P_1^{(2)}}{a_1 + P_1^{(2)}} - \varepsilon - r_2 Z^{(2)} & 0 \\ -\frac{k P_1^{(2)}}{(a_1 + N^{(2)})^2} & \frac{k}{a_1 + N^{(2)}} & 0 & 0 & -\theta \end{bmatrix}$$

The characteristic equation of this Jacobian matrix is given by

$$\left(-\gamma + \frac{m_2 N^{(2)}}{a_2 + N^{(2)}} - r_1 Z^{(2)} - \lambda_{2P_2}\right)\left(-\varepsilon + \frac{\beta_1 k_1 P_1^{(2)}}{a_1 + P_1^{(2)}} - r_2 Z^{(2)} - \lambda_{2P_3}\right)(-\theta - \lambda_{2Z})(\lambda_2^2 + A_1\lambda_2 + A_2) = 0$$

Therefore,

$$\lambda_{2P_1,2N} = \frac{-dw - \delta w + y + m_1 N^{(2)^2}}{2w} \pm \sqrt{(dw + \delta w - y)^2 - 4w\left(d\delta w - \delta y - dm_1 N^{(2)^2}\right)}$$

where

$$w = a_1^2 + 2a_1 N^{(2)} + N^{(2)^2}$$

$$y = a_1 m_1 N^{(2)} - a_1 m_1 P_1^{(2)}$$

Since all the eigenvalues have negative real parts, then the equilibrium point $E_2$ is locally asymptotically stable if the existence condition satisfies the following conditions:

$$\frac{m_2 N^{(2)}}{a_2 + N^{(2)}} < \gamma + r_1 Z^{(2)}$$

$$\frac{\beta_1 k_1 P_1^{(2)}}{a_1 + P_1^{(2)}} < \varepsilon + r_2 Z^{(2)}$$

## Biological interpretation

When the amount of nutrient and toxic chemicals are stabilized at $N^{(2)}$ and $Z^{(2)}$,respectively, NTP and zooplankton populations become extinct if their maximal growth rates $\frac{m_2 N^{(2)}}{a_2 + N^{(2)}}$ and $\frac{\beta k_1 P_1^{(2)}}{a_1 + P_1^{(2)}}$ are less than total loss rate $\gamma + r_1 Z^{(2)}$ and $\varepsilon + r_2 Z^{(2)}$.

The Jacobian matrix at $E^* = (N^*, P_1^*, P_2^*, P_3^*, Z^*)$ is

$$J(N^*, P_1^*, P_2^*, P_3^*, Z^*) = \begin{bmatrix} A_1 & A_2 & A_3 & 0 & 0 \\ A_4 & A_5 & 0 & A_6 & 0 \\ A_7 & 0 & A_8 & A_9 & -P_2^* r_1 \\ 0 & A_{10} & A_{11} & A_{12} & -P_3^* r_2 \\ A_{13} & A_{14} & 0 & 0 & -\theta \end{bmatrix}$$

where

$$A_1 = -d + \frac{m_1 N^* P_1^*}{(a_1 + N^*)^2} - \frac{m_1 P_1^*}{a_1 + N^*} + \frac{m_2 N^* P_2^*}{(a_2 + N^*)^2} - \frac{m_2 P_2^*}{a_2 + N^*}$$

$$A_2 = -\frac{m_1 N^*}{a_1 + N^*}$$

$$A_3 = -\frac{m_2 N^*}{a_2 + N^*}$$

$$A_4 = -\frac{m_1 N^* P_1^*}{(a_1 + N^*)^2} + \frac{m_1 P_2^*}{a_1 + N^*}$$

$$A_5 = -\delta + \frac{m_1 N^*}{a_1 + N^*} + \frac{\beta_1 P_1^* P_3^*}{(a_1 + P_1^*)^2} - \frac{\beta_1 P_3^*}{a_1 + P_1^*}$$

$$A_6 = -\frac{\beta_1 P_1^*}{a_1 + P_1^*}$$

$$A_7 = -\frac{m_2 N^* P_2^*}{(a_2 + N^*)^2} + \frac{m_2 P_2^*}{a_2 + N^*}$$

$$A_8 = -\gamma + \frac{m_2 N^*}{a_2 + N^*} + \frac{\beta_2 P_2^* P_3^*}{(a_2 + P_2^*)^2} - \frac{\beta_2 P_3^*}{a_2 + P_2^*} - r_1 Z^*$$

$$A_9 = -\frac{\beta_2 P_2^*}{a_2 + P_2^*}$$

$$A_{10} = -\frac{\beta_1 k_1 P_1^* P_3^*}{(a_1 + P_1^*)^2} + \frac{\beta_1 k_1 P_3^*}{a_1 + P_1^*}$$

$$A_{11} = -\frac{\beta_2 k_2 P_2^* P_3^*}{(a_2 + P_2^*)^2} + \frac{\beta_2 k_2 P_3^*}{a_2 + P_2^*}$$

$$A_{12} = \frac{\beta_1 k_1 P_1^*}{a_1 + P_1^*} + \frac{\beta_2 k_2 P_2^*}{a_2 + P_2^*} - \varepsilon - r_2 Z^*$$

$$A_{13} = -\frac{k P_1^*}{(a_1 + N^*)^2}$$

$$A_{14} = \frac{k}{a_1 + N^*}$$

$$\det|J - \lambda I| = \begin{vmatrix} A_1 - \lambda & A_2 & A_3 & 0 & 0 \\ A_4 & A_5 - \lambda & 0 & A_6 & 0 \\ A_7 & 0 & A_8 - \lambda & A_9 & -P_2^* r_1 \\ 0 & A_{10} & A_{11} & A_{12} - \lambda & -P_3 r_2 \\ A_{13} & A_{14} & 0 & 0 & -\theta - \lambda \end{vmatrix} = 0$$

## Characteristic equation

$$\lambda^5 + B_1\lambda^4 + B_2\lambda^3 + B_3\lambda^2 + B_4\lambda + B_5 = 0$$

where

$$B_1 = -A_1 - A_3 - A_5 - A_8 + \theta$$

$$B_2 = A_1A_3 - A_{10}A_4 + A_1A_5 + A_3A_5 - A_{11}A_6 - A_{12}A_7 + A_1A_8 + A_3A_8 + A_5A_8 - A_2A_9 - A_1\theta - A_3\theta - A_5\theta - A_8\theta$$

$$B_3 = A_{10}A_3A_4 - A_1A_3A_5 + A_1A_{11}A_6 + A_{11}A_5A_6 + A_1A_{12}A_7 + A_{12}A_3A_7 - A_1A_3A_8 + A_{10}A_4A_8 - A_1A_5A_8 - A_3A_5A_8 + A_2A_5A_9 + A_2A_8A_9 + A_1A_3\theta - A_{10}A_4\theta + A_1A_5\theta + A_3A_5\theta - A_{11}A_6\theta - A_{12}A_7\theta + A_1A_8\theta + A_3A_8\theta + A_5A_8\theta - A_2A_9\theta + A_{10}A_{13}P_2^* r_1 + A_{11}A_{14}P_3^* r_2$$

$$B_4 = -A_{10}A_{12}A_2A_6 + A_{10}A_{11}A_4A_6 - A_1A_{11}A_5A_6 - A_1A_{12}A_3A_7$$
$$- A_{10}A_3A_4A_8 + A_1A_3A_5A_8 + A_{12}A_2A_7A_9 - A_{11}A_4A_7A_9$$
$$- A_2A_5A_8A_9 + A_{10}A_3A_4\theta - A_1A_3A_5\theta + A_1A_{11}A_6\theta + A_{11}A_5A_6\theta$$
$$+ A_1A_{12}A_7\theta + A_{12}A_3A_7\theta - A_1A_3A_8\theta + A_{10}A_4A_8\theta - A_1A_5A_8\theta$$
$$- A_3A_5A_8\theta + A_2A_5A_9\theta + A_2A_8A_9\theta + A_2A_8A_9\theta + A_{10}A_{14}A_2P_2^*r_1$$
$$- A_{10}A_{13}A_3P_2^*r_1 + A_{11}A_{14}A_7P_2^*r_1 - A_{10}A_{13}A_8P_2^*r_1 + A_{10}A_{12}A_{13}P_3^*r_2$$
$$- A_1A_{11}A_{14}P_3^*r_2 - A_{11}A_{14}A_5P_3^*r_2 + A_{11}A_{13}A_9P_3^*r_2$$

$$B_5 = -A_{10}A_{12}A_2A_6\theta + A_{10}A_{11}A_4A_6\theta - A_1A_{11}A_5A_6\theta$$
$$- A_1A_{12}A_3A_7\theta - A_{10}A_3A_4A_8\theta + A_1A_3A_5A_8\theta + A_{12}A_2A_7A_9\theta$$
$$- A_{11}A_4A_7A_9\theta - A_2A_5A_8A_9\theta - A_{10}A_{11}A_{13}A_6P_2^*r_1$$
$$- A_1A_{11}A_{14}A_7P_2^*r_1 - A_{10}A_{14}A_2A_8P_2^*r_1 - A_{10}A_{14}A_2A_8P_2^*r_1$$
$$+ A_{10}A_{13}A_3A_8P_2^*r_1 + A_{10}A_{13}A_3A_8P_2^*r_1 + A_{11}A_{13}A_7A_9P_2^*r_1$$
$$+ A_{10}A_{12}A_{14}A_2P_3^*r_2 - A_{10}A_{12}A_{13}A_3P_3^*r_2 - A_{10}A_{11}A_{14}A_4P_3^*r_2$$
$$+ A_1A_{11}A_{14}A_5P_3^*r_2 - A_{11}A_{13}A_5A_9P_3^*r_2$$

A system is stable if all the eigenvalues are negative real parts. Thus, by following Routh-Hurwitz criterion, all the eigenvalues must satisfy the following conditions:

$$B_i > 0$$
$$B_1B_2B_3 > B_3^2 + B_1^2B_4$$
$$(B_1B_4 - B_5)(B_1B_2B_3 - B_3^2 - B_1^2B_4) > B_5(B_1B_2 - B_3)^2 + B_1B_5^2$$

**Biological interpretation**

If all the roots satisfy the above conditions, then the equilibrium point of $E^*$ is stable, which indicates that there is no occurrence of HABs.

## NUMERICAL SIMULATIONS

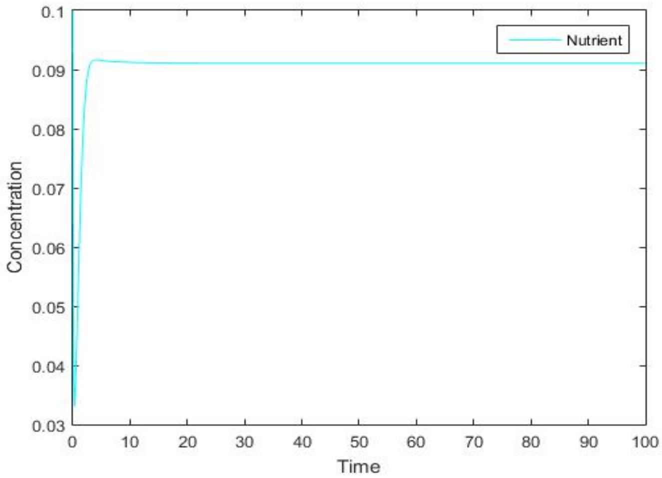
In this section, numerical simulations for system (1) is performed to study the dynamical behavior of the system. By varying the parameter value of $A$, and the other values of parameter are as shown in Table 1. The simulation results are obtained and it is observed that, system (1) has an equilibrium point at $E^* = (N^*, P_1^*, P_2^*, P_3^*, Z^*)$ as shown in Figure 1 when the parameter value is $A$=1.0.

**Table 1** Parameter values used in numerical analysis.

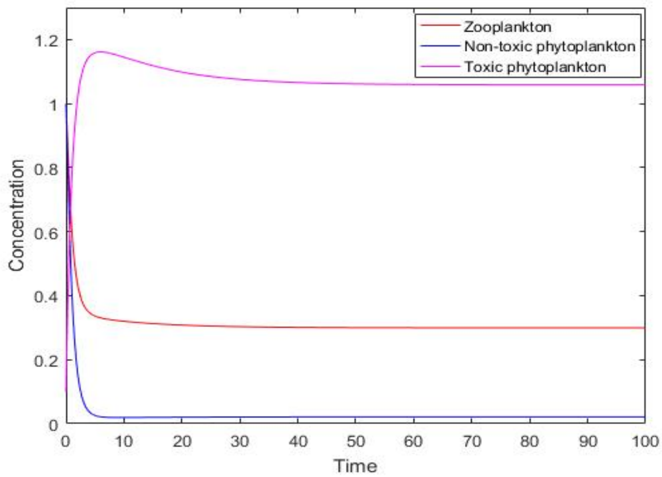
| Name | Description | Values |
|---|---|---|
| $d$ | Amount of nutrient loss | 0.1 |
| $a_1$ | Half saturation constant | 0.07 |
| $a_2$ | Half saturation constant | 0.2 |
| $m_1$ | Uptake rate of TPP | 1.0 |
| $m_2$ | Uptake rate of NTP | 0.7 |
| $k$ | Toxin production rate | 0.07 |
| $k_1$ | Conversion rate of TPP | 0.01 |
| $k_2$ | Conversion rate of NTP | 0.01 |
| $r_1$ | Mortality rate of NTP due to toxin | 0.8 |
| $r_2$ | Mortality rate of zooplankton due to toxin | 0.8 |
| $\varepsilon$ | Per capita natural death rate of zooplankton | 0.02 |
| $\beta_1$ | Maximum zooplankton ingestion rate on TPP | 0.01 |
| $\beta_2$ | Maximum zooplankton ingestion rate on NTP | 0.02 |
| $\theta$ | Washout rate of toxin | 0.8 |
| $\gamma$ | Per capita natural death rate of NTP | 0.01 |
| $\delta$ | Per capita natural death rate of TPP | 0.5 |

If the parameter value is $A$=1.5 and the other values of parameter are as shown in Table 1, hence system (1) has an equilibrium point at $E_1=(N^{(1)}, 0,0,0,0)$ as shown in Figure 2. Next, if we consider parameter value of $A$=0.5 and the other values of parameter are as shown in Table 1, then system (1) has an equilibrium point at $E_2=(N^{(2)}, P_1^{(2)}, 0, 0, Z^{(2)})$ as shown in Figure 3. Table 1 shows the parameter values used in this model [4].

a) Nutrient.




b) Toxic phytoplankton, non-toxic phytoplankton, and zooplankton.




c) Toxic chemicals.

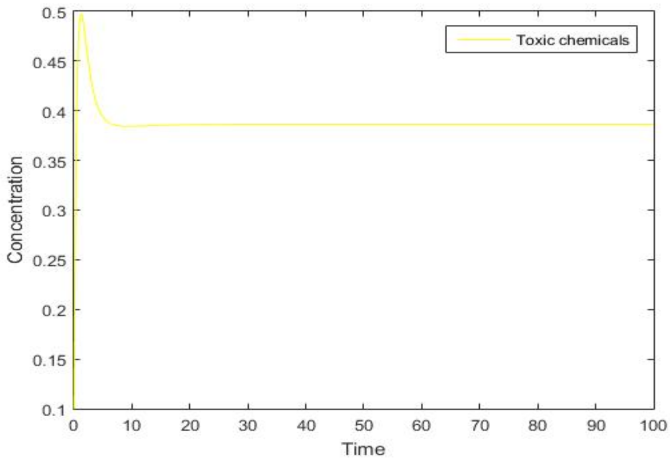
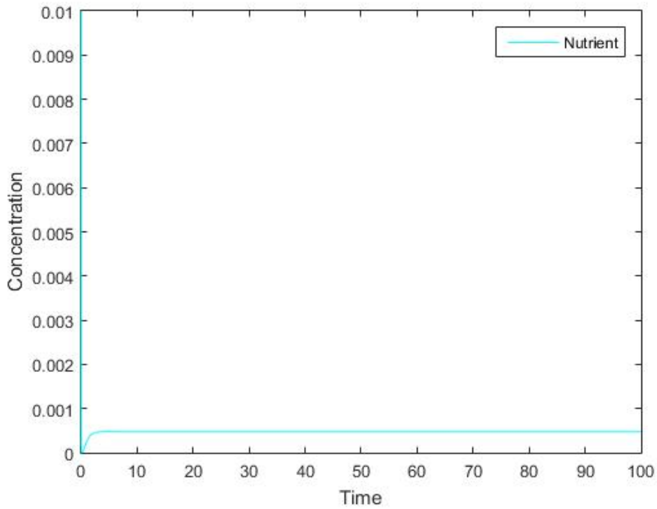
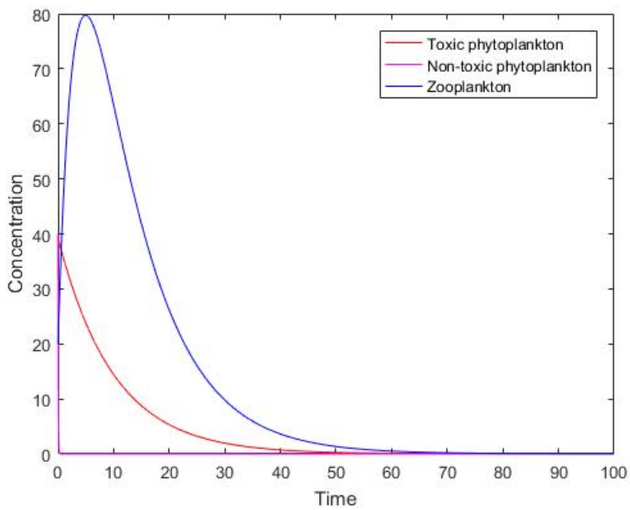



**Figure 1** From (a-e), the solution goes to components $E_2=(N^*, P_1^*, P_2^*, P_3^*, Z^*)$ when the parameter value is $A = 1.0$.

a) Nutrient




b) Toxic phytoplankton, non-toxic phytoplankton, and zooplankton.




c) Toxic chemicals.

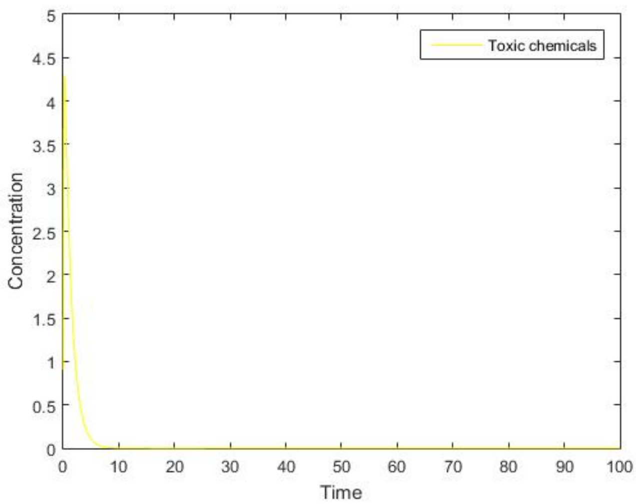
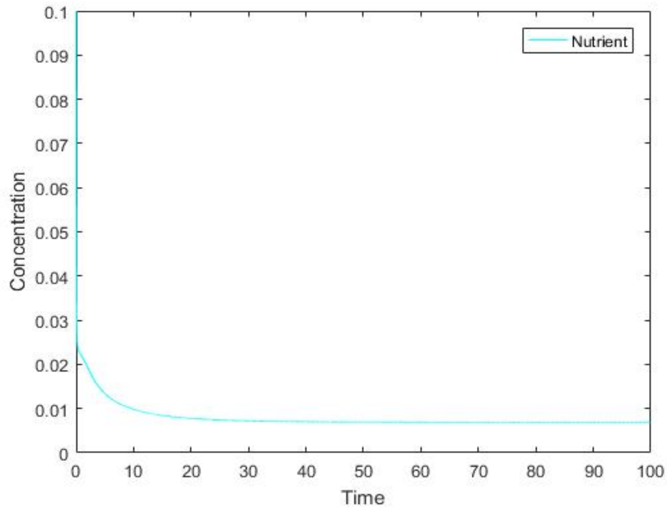
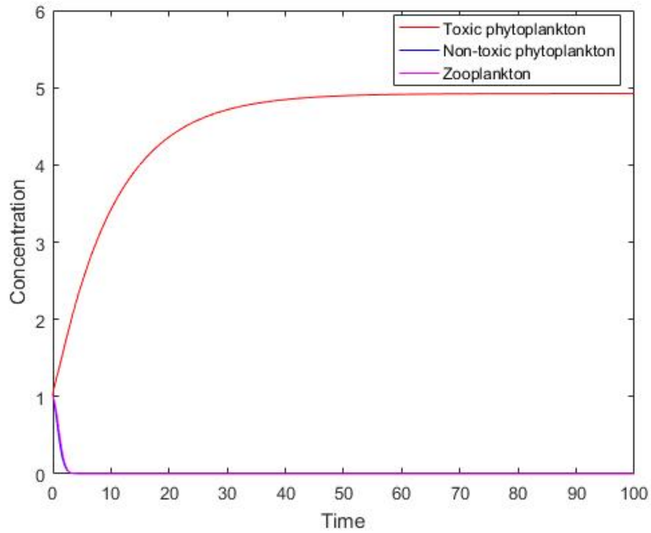



**Figure 2** From (a-e), the solution goes to components $E_1$=($N^{(1)}$, 0,0,0,0) where the parameter value is $A$ = 1.5.

a) Nutrient




b) Toxic phytoplankton, non-toxic phytoplankton, and zooplankton.




c) Toxic chemicals.

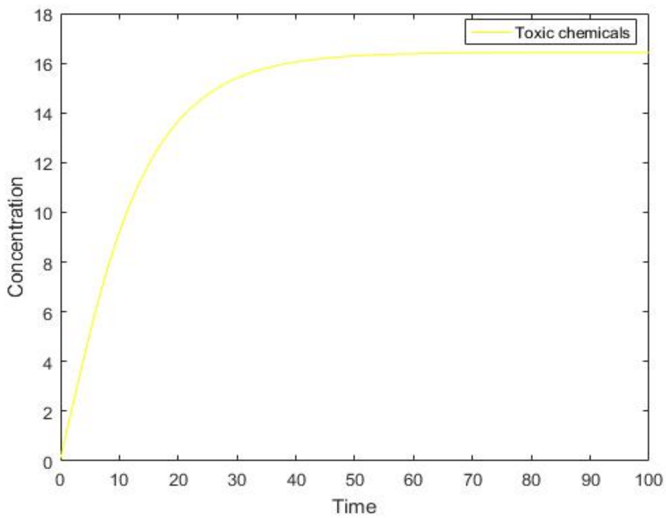



**Figure 3** From (a-e), the solution goes to components $E_2$=($N^{(2)}$, $P_1^{(2)}$, 0, 0, $Z^{(2)}$) where the parameter value is $A$ = 0.5.

## DISCUSSION

In Figure 1, the solution is asymptotically stable at component $E_2=(N^*, P_1^*, P_2^*, P_3^*, Z^*)$ where the parameter value is $A=1.0$. In this situation, HAB does not occur as all the variables exist in the ecosystem and the system is balance. Next, in Figure 2, it can be seen that, when the parameter value of $A=1.5$, the solution goes to components $E_1=(N^{(1)},0,0,0,0)$ where in this situation, only nutrient is present in the water. Therefore, HAB will not occur in this context.

In Figure 3, when the parameter value is $A=0.5$, the solution is asymptotically stable at components $E^*=(N^*,P_1^*,0,0,Z^*)$.From the graph, it can be seen that when the nutrient is limited, TPP produces more toxic chemicals and there is no significant change when the nutrient is sufficient. This shows that the environmental stress forced the TPP to produce more toxic chemicals. As the amount of toxic chemicals increase in the water, concentrations of NTP and zooplankton drop tremendously. Hence, it shows that HAB caused other population to die due to the effects of toxic chemicals by TPP in the water.

## CONCLUSION

The results obtained demonstrate that when the nutrient is limited, TPP releases more toxic chemicals. This is due to the environmental stress faced by the TPP population since the nutrient is limited forcing them to release more toxic chemicals. When toxic chemicals concentration is higher in the water, this situation affects the co-existence of NTP and zooplankton populations. Toxic chemicals produced by TPP population will harm the NTP and zooplankton populations, causing these populations to cease. Elimination of zooplankton population as the grazer in the system results in ecosystem imbalance, thus allowing the TPP population to dominate the ecosystem. Therefore, the amount of toxic chemicals in the water will increase and this will pose adverse effect to human and also marine organism. Hence, the existence of zooplankton population is important to balance the marine ecosystem as well as to control the algal bloom. The data shown from the model presented enable us to have better understanding on the occurrence of HAB and the importance of population dynamic to prevent ecosystem imbalance.

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