MJFAS MALAYSIAN JOURNAL OF FUNDAMENTAL AND APPLIED SCIENCES
PRINT ISSN: 2289-5981 | ONLINE ISSN: 2289-599X

RESEARCH ARTICLE

The depolarization factors for ellipsoids and some of their properties

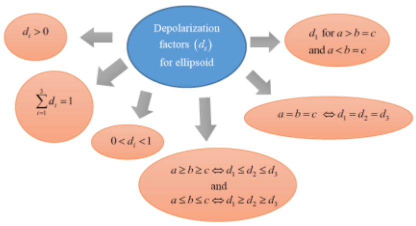
Nurhazirah Mohamad Yunos [a], Taufiq Khairi Ahmad Khairuddin [a, b,*], Sharidan Shafie [a], Tahir Ahmad [a, c], William Lionheart [d]

[a] Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor, Malaysia
[b] UTM Centre for Industrial and Applied Mathematics (UTM-CIAM), Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor, Malaysia
[c] Centre for Sustainable Nanomaterials, Ibnu Sina Institute for Scientific and Industrial Research, Universiti Teknologi Malaysia, Skudai 81310, Johor, Malaysia
[d] School of Mathematics, The University of Manchester, M13 9PL Manchester, United Kingdom

* Corresponding author: taufiq@utm.my

Article history
Received 19 December 2018
Revised 13 February 2019
Accepted 27 February 2019
Published Online 3 December 2019

Graphical abstract



Abstract

The terminology depolarization factors was firstly highlighted in the study of problems involving magnetic, where, it was initially used to describe magnetic properties of material. Recently, this terminology was investigated to describe composites, improve imaging techniques, and other field of researches related to potential theory in mathematics and physics. Due to our interest in electrical imaging using polarization tensor (PT) and since PT is actually related to the depolarization factors, in this paper, some properties of the depolarization factors are investigated for future applications. The values of these depolarization factors are firstly proven to be non-negative. Based on the previous studies which consider the incomplete elliptic integrals of the first and second kind with some suitable identities, the summation of the depolarization factors are shown to be equal to one. By using these two properties, the value for each depolarization factor for ellipsoid is then explained to be between zero and one. It is also shown in this paper that the depolarization factors can be characterized based on the values of the semi principal axes of the ellipsoid. Reversely, the semi principal axes of the ellipsoid can be classified based on the values of the depolarization factors. All properties presented in this paper could be useful and important in the future especially to use the depolarization factors in any related applications.

Keywords: Elliptic integrals, polarization tensor, electromagnetism

INTRODUCTION

Depolarization factors (also known as demagnetizing factors) are widely used in potential problems. When a material of an irregular shape is magnetized by a uniformly applied field, the total field inside the material and its surrounding change. There might be some difficulty in measuring the magnetivity of the material due to the magnetization. Thus, depolarization factors were studied to correct the data on certain magnetic material [1]. Previously, the problems of potential for an ellipsoid were investigated by Maxwell [2]. Maxwell [2] derived the formula of the depolarization factors for ellipsoid from a definite integral given by [3] where the depolarization factors were actually defined and labeled as L, M, and N. In 1945, Osborn [4] studied the depolarization factors of the general ellipsoid but gave a slightly different formula of the depolarization factors by using the elliptic integral expressions. In [4], the depolarization factors were scalled in terms of $L/4\pi$, $M/4\pi$ and $N/4\pi$.

In 1945, Stoner [5] introduced the formula of the depolarization factors for ellipsoids as integral equations in terms of modified factors, D_a, D_b and D_c, not L, M and N. In addition, some derivations of the depolarization factors for sphere and spheroids (prolate and oblate shperoid) were presented here. Besides, there are also detailed discussion about depolarization factors for a few other geometries such as rectangle [1, 6] and cylinder [7]. In spite of all previous geometries mentioned, by using the method as suggested in [8], any complicated shapes can be equivalent to ellipsoid by setting their depolarization factors to be equal. Furthermore, Milton [9] used depolarization factors to study composite and he also showed some useful properties of depolarization factors, where the properties are reviewed in the following section.

In another development, the Polarization Tensor (PT) has been widely explored for many purposes especially in electric and electromagnetism. Recently, PT has been used in the real applications such as electrical imaging [10], metal detection (for security screening [11,12], and landmine clearance [13]) and also electrosensing fish [14,15,16,17]. This terminology actually originates from the study of virtual mass for examples in [18,19]. Generally, PT is used to describe the the perturbation in electric [10, 20,21,22] and electromagnetic fields [23,24,25,26,27] that occur due to the conductivity contrast between a conducting object and a free space such as $\mathbb{R}^2$ or $\mathbb{R}^3$. Specifically, PT can also describe the conducting object itself [10] and thus, it is commonly referred as PT for the conducting object. Here, PT can be computed by using the explicit formula or also by making field measurements in the laboratory as well as during a field work. Some examples showing the computation of PT based on the explicit formula were given in [26, 27,28,29,30,31,32] whereas [11,12,13] described how to determine PT based on field measurements.

Due to its significant in those applications, there are many researches that discuss the properties of the PT [10,22,24, 26,27,28,31,32,33,34,35]. These properties are important as an aid to classify the objects based on their PT. Moreover, some investigations had adapted the depolarization factors for ellipsoid to reveal the properties of ellipsoid based on its PT. For example, Mohamad Yunos and Ahmad Khairuddin [34,35] investigated the PT for spheroid by adapting the depolarization factors into the analytical formula of the PT for ellipsoid. Besides, the findings in Ahmad Khairuddin et al. [33] showed that the conductivity and material of a spheroid can be classified according to its PT. Therefore, due to their relationship with our current studies about PT, some properties of the depolarization factors specifically for ellipsoid are revised and determined in this paper.

FORMULATIONS AND RESULTS

First, we review the mathematical formulation for the depolarization factors for ellipsoid. If an ellipsoid is placed in a uniform applied magnetic field (denoted as H_a), the magnetization (denoted as M) and demagnetizing field (denoted as H_d) are both uniform [2] and are related by

$$H_d = H_a - d_i M$$

where d_i for $i = 1,2,3$ are called as the depolarization factors for the ellipsoid in the x, y and z directions. Let a, b and c be the semi principal axes of an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. The definitions of depolarization factors for ellipsoid, the triplet (d_1, d_2, d_3) given by Milton [9] as follows:

$$d_1 = \frac{abc}{2}\int_0^\infty \frac{dy}{\left(y+a^2\right)^{3/2}\sqrt{\left(y+b^2\right)\left(y+c^2\right)}}, \tag{1}$$

$$d_2 = \frac{abc}{2}\int_0^\infty \frac{dy}{\left(y+b^2\right)^{3/2}\sqrt{\left(y+a^2\right)\left(y+c^2\right)}}, \tag{2}$$

$$d_3 = \frac{abc}{2}\int_0^\infty \frac{dy}{\left(y+c^2\right)^{3/2}\sqrt{\left(y+a^2\right)\left(y+b^2\right)}}. \tag{3}$$

Equation (1), (2) and (3) are used in this paper when describing depolarization factors for ellipsoid.

Theorem 1 The depolarization factors (d_1, d_2, d_3) for ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ each is a positive number.

Proof Since $a,b,c > 0$, obviously $\frac{abc}{2} > 0$. Now, in order to show $d_i > 0$ for $i = 1,2,3$, we have to show that the improper integrals in (1), (2) and (3) are all positive. The integrands $\frac{1}{\left(y+a^2\right)^{3/2}\sqrt{\left(y+b^2\right)\left(y+c^2\right)}}$, $\frac{1}{\left(y+b^2\right)^{3/2}\sqrt{\left(y+a^2\right)\left(y+c^2\right)}}$ and $\frac{1}{\left(y+c^2\right)^{3/2}\sqrt{\left(y+a^2\right)\left(y+b^2\right)}}$ are continuous and positive for $y \in [0,\infty)$ and $a,b,c > 0$. This suggests that the corresponding integrals are also positive. Thus, in order to show the integrals are positive, we must show that the integrals converge.

We use comparison test in this proving. First, let $\left(y+b^2\right)\left(y+c^2\right) \geq b^2c^2$ since $y \in [0,\infty)$. Then,

$$\sqrt{\left(y+b^2\right)\left(y+c^2\right)} \geq \sqrt{b^2c^2},$$

$$\frac{1}{\sqrt{\left(y+b^2\right)\left(y+c^2\right)}} \leq \frac{1}{\sqrt{b^2c^2}}.$$

So,

$$0 < \int_0^\infty \frac{abc\ dy}{2\left(y+a^2\right)^{3/2}\sqrt{\left(y+b^2\right)\left(y+c^2\right)}} \leq \int_0^\infty \frac{abc\ dy}{2\left(y+a^2\right)^{3/2}\sqrt{b^2c^2}},$$

and from (1),

$$0 < d_1 \leq \int_0^\infty \frac{abc\ dy}{2\left(y+a^2\right)^{3/2}\sqrt{b^2c^2}} = 1.$$

Similarly, by letting $\left(y+a^2\right)\left(y+c^2\right) \geq a^2c^2$, we may have

$$\sqrt{\left(y+a^2\right)\left(y+c^2\right)} \geq \sqrt{a^2c^2},$$

$$\frac{1}{\sqrt{\left(y+a^2\right)\left(y+c^2\right)}} \leq \frac{1}{\sqrt{a^2c^2}}.$$

Thus,

$$0 < \int_0^\infty \frac{abc\ dy}{2\left(y+b^2\right)^{3/2}\sqrt{\left(y+a^2\right)\left(y+c^2\right)}} \leq \int_0^\infty \frac{abc\ dy}{2\left(y+b^2\right)^{3/2}\sqrt{a^2c^2}}.$$

From (2), we have

$$0 < d_2 \leq \int_0^\infty \frac{abc\ dy}{2\left(y+b^2\right)^{3/2}\sqrt{a^2c^2}} = 1.$$

Also,

$\left(y+a^2\right)\left(y+b^2\right) \geq a^2b^2$ gives

$$\sqrt{\left(y+a^2\right)\left(y+b^2\right)} \geq \sqrt{a^2b^2},$$

$$\frac{1}{\sqrt{\left(y+a^2\right)\left(y+b^2\right)}} \leq \frac{1}{\sqrt{a^2b^2}}.$$

Hence,

$$0 < \int_0^\infty \frac{abc\ dy}{2\left(y+c^2\right)^{3/2}\sqrt{\left(y+a^2\right)\left(y+b^2\right)}} \leq \int_0^\infty \frac{abc\ dy}{2\left(y+c^2\right)^{3/2}\sqrt{a^2b^2}},$$

and from (3),

$$0 < d_3 \leq \int_0^\infty \frac{abc\ dy}{2\left(y+c^2\right)^{3/2}\sqrt{a^2b^2}} = 1.$$

By comparison test, each integral converges and therefore, $d_1, d_2, d_3 > 0.$

Theorem 1 and its proof support the claim on page 133 by Milton [9]. Next, we reprove the next property which has been given in Stoner [5] and Milton [9]. In our proof, a different identity is used to simplify d_2.

Proposition 2 The depolarization factors (d_1, d_2, d_3) satisfy $d_1 + d_2 + d_3 = 1.$

Proof In our proof, we will use the incomplete elliptic integrals of the first and second kind given as below,

$$F(k,\phi)=\int_0^x \frac{dz}{\sqrt{\left(1-z^2\right)\left(1-k^2z^2\right)}}\,, \tag{4}$$

$$E(k,\phi)=\int_0^x \frac{dz\sqrt{\left(1-k^2z^2\right)}}{\sqrt{\left(1-z^2\right)}}\,, \tag{5}$$

where,

$$k=\sqrt{\frac{\left(a^2-b^2\right)}{\left(a^2-c^2\right)}},\qquad x=\sqrt{1-\frac{c^2}{a^2}}\ .$$

In order to use (4) and (5), we transform (1), (2) and (3) first. By using the substitution $z^2=\left(a^2-c^2\right)/\left(a^2+y\right)$ as given by Stoner [5], (1), (2) and (3) become

$$d_1=\frac{abc}{\left(a^2-c^2\right)^{3/2}}\int_0^x \frac{z^2dz}{\sqrt{\left(1-z^2\right)\left(1-k^2z^2\right)}}, \tag{6}$$

$$d_2=\frac{abc}{\left(a^2-c^2\right)^{3/2}}\int_0^x \frac{z^2dz}{\sqrt{\left(1-z^2\right)\left(1-k^2z^2\right)^3}}, \tag{7}$$

$$d_3=\frac{abc}{\left(a^2-c^2\right)^{3/2}}\int_0^x \frac{z^2dz}{\sqrt{\left(1-z^2\right)^3\left(1-k^2z^2\right)}}. \tag{8}$$

By using the following identity

$$\frac{z^2}{\sqrt{\left(1-z^2\right)\left(1-k^2z^2\right)}}=\frac{1}{k^2}\left\{\frac{1}{\sqrt{\left(1-z^2\right)\left(1-k^2z^2\right)}}-\frac{\sqrt{\left(1-k^2z^2\right)}}{\left(1-z^2\right)}\right\} \tag{9}$$

in (6) , d_1 can be expressed as

$$d_1=\frac{abc(F-E)}{(a^2-b^2)\sqrt{(a^2-c^2)}}. \tag{10}$$

The next relation

$$\int_0^x \frac{d}{dz}\left\{\frac{z\sqrt{\left(1-z^2\right)}}{\sqrt{\left(1-k^2z^2\right)}}\right\}dz=\int_0^x \frac{\sqrt{\left(1-z^2\right)}}{\sqrt{\left(1-k^2z^2\right)}}dz-\int_0^x \frac{(1-k^2)z^2}{\sqrt{\left(1-z^2\right)\left(1-k^2z^2\right)^3}}dz \tag{11}$$

and the identity

$$\frac{\sqrt{\left(1-z^2\right)}}{\sqrt{\left(1-k^2z^2\right)}}=\frac{\sqrt{\left(1-k^2z^2\right)}}{\sqrt{\left(1-z^2\right)}}-\frac{(1-k^2)z^2}{\sqrt{\left(1-z^2\right)\left(1-k^2z^2\right)}} \tag{12}$$

can be used in (7) to get the expression for d_2 in terms of (4) and (5) as follows

$$d_2=\frac{abc(E-F)}{(a^2-b^2)\sqrt{(a^2-c^2)}}+\frac{abcE}{(b^2-c^2)\sqrt{(a^2-c^2)}}-\frac{c^2}{b^2-c^2}. \tag{13}$$

Besides, the term for d_3 can be obtained by using the relation below

$$\int_0^x \frac{d}{dz}\left\{\frac{z\sqrt{\left(1-k^2z^2\right)}}{\sqrt{\left(1-z^2\right)}}\right\}dz=\int_0^x \frac{\sqrt{\left(1-k^2z^2\right)}}{\sqrt{\left(1-z^2\right)}}dz-\int_0^x \frac{(1-k^2)z^2}{\sqrt{\left(1-z^2\right)^3\left(1-k^2z^2\right)}}dz \tag{14}$$

in (8) where

$$d_3=\frac{b^2}{b^2-c^2}-\frac{abcE}{(b^2-c^2)\sqrt{(a^2-c^2)}}. \tag{15}$$

Now, we sum up (10), (13) and (15) to obtain

$$\begin{aligned} d_1+d_2+d_3 &= -\frac{c^2}{b^2-c^2}+\frac{b^2}{b^2-c^2}, \\ &= 1. \end{aligned}$$

□

Proposition 2 shows that for any value of depolarization factors, the summation of the depolarization factors must always equal to 1. By using the information in Theorem 1 and Proposition 2, we propose another property of the depolarization factors in the next theorem.

Theorem 3 The depolarization factors $\left(d_1,d_2,d_3\right)$ each satisfy $0<d_i<1$ for $i=1,2,3.$

Proof First of all, we consider d_1 where we want to show that $0<d_i<1.$ This can be proven by contradiction. Assume that $0<d_1<1$ is false as $0\geq d_1$ or $d_1\geq 1.$ Obviously, $d_1\leq 0$ is a contradiction since $d_1>0$ according to Theorem 1. So, $d_1>0.$

Next, if $d_1\geq 1$, from Proposition 2, we obtain $1-d_2-d_3\geq 1.$ So, $0\geq d_2-d_3.$ This is a contradiction since $d_2,d_3>0$ according to Theorem 1. Hence, $d_1<1.$

Therefore, since $d_1>0$ and $d_1<1$ from both cases, we have proved that $0<d_1<1.$ The same steps can be repeated for d_2 and d_3. Hence, it is proven that $0<d_i<1$ for $i=1,2,3.$ □

According to Theorem 3, the values for the depolarization factors are between 0 and 1. Many examples to numerically justify Theorem 3 can be found in [5] and [9]. The next theorem relates semi principal axes of an ellipsoid and its depolarization factors.

Theorem 4 Let a,b and c be the semi principal axes of ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and d_i is the depolarization factors for $i=1,2,3.$

i. $a\geq b\geq c$ if and only if $d_1\leq d_2\leq d_3.$
ii. $a\leq b\leq c$ if and only if $d_1\geq d_2\geq d_3.$

Proof We prove part (i) as similar steps can be used to prove part (ii). Assume $a\geq b\geq c$. We want to show $d_1\leq d_2\leq d_3$**.** First, we consider $a\geq b$. We must show $d_1\leq d_2$. We have

$$\begin{gathered} a\geq b, \\ a^2\geq b^2, \\ y+a^2\geq y+b^2, \\ \frac{1}{y+a^2}\leq\frac{1}{y+b^2}, \end{gathered}$$

$$\frac{abc}{2}\int_0^\infty \frac{dy}{\left(y+a^2\right)^{3/2}\sqrt{\left(y+b^2\right)\left(y+c^2\right)}}\leq\frac{abc}{2}\int_0^\infty \frac{dy}{\left(y+b^2\right)^{3/2}\sqrt{\left(y+a^2\right)\left(y+c^2\right)}},$$

which implies $d_1\leq d_2$ from (1) and (2).

Now, suppose $b\geq c$ and we will show that $d_2\leq d_3.$ Since $b\geq c,$

$$\begin{gathered} b^2\geq c^2, \\ y+b^2\geq y+c^2, \\ \frac{1}{y+b^2}\leq\frac{1}{y+c^2}, \end{gathered}$$

$$\frac{abc}{2}\int_0^\infty \frac{dy}{\left(y+b^2\right)^{3/2}\sqrt{\left(y+a^2\right)\left(y+c^2\right)}}\leq\frac{abc}{2}\int_0^\infty \frac{dy}{\left(y+c^2\right)^{3/2}\sqrt{\left(y+a^2\right)\left(y+b^2\right)}},$$

which also indicates that $d_2 \le d_3$ from (2) and (3). By combining all the proof, it is proven that $d_1 \le d_2 \le d_3$ if $a \ge b \ge c$.

Next, we assume $d_1 \le d_2 \le d_3$ and we want to show that $a \ge b \ge c$. This can be proven by contrapositive. Assume $a < b$ or $b < c$ and we want to show $d_1 > d_2$ or $d_2 > d_3$, respectively. Since $a < b$, we have

$$a^2 < b^2,$$
$$y + a^2 < y + b^2,$$
$$\frac{1}{y+a^2} > \frac{1}{y+b^2},$$
$$\frac{abc}{2}\int_0^\infty \frac{dy}{(y+a^2)^{3/2}\sqrt{(y+b^2)(y+c^2)}} > \frac{abc}{2}\int_0^\infty \frac{dy}{(y+b^2)^{3/2}\sqrt{(y+a^2)(y+c^2)}},$$

which leads to $d_1 > d_2$.

Now, assume $b < c$. So, we have

$$b^2 < c^2,$$
$$y + b^2 < y + c^2,$$
$$\frac{1}{y+b^2} > \frac{1}{y+c^2},$$
$$\frac{abc}{2}\int_0^\infty \frac{dy}{(y+b^2)^{3/2}\sqrt{(y+a^2)(y+c^2)}} > \frac{abc}{2}\int_0^\infty \frac{dy}{(y+c^2)^{3/2}\sqrt{(y+a^2)(y+b^2)}}$$

which imply that $d_2 > d_3$. As the contrapositive has been proven true, this means the original statement is also true, which is if $d_1 \le d_2 \le d_3$ then $a \ge b \ge c$.

Based on Theorem 4, we can predict the range among the depolarization factors based on the semi principal axes of the ellipsoid. Reversely, we can describe the semi principal axes of the ellipsoid based on the values of the depolarization factors. In this theorem, the value of the semi principal axes and the depolarization factors can be reordered accordingly. For example, $c \ge a \ge b$ if and only if $d_3 \le d_1 \le d_2$. Furthermore, based on this theorem and considering only equality and not inequality, we can have the next two corollaries.

Corollary 5 Let a,b and c be the semi principal axes of ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and d_i is the depolarization factors for $i=1,2,3$. $a=b=c$ if and only if $d_1=d_2=d_3$.

Proof Assume $a=b=c$. Consider $a=b$. We need to show $d_1=d_2$. By substituting $a=b$ into both (1) and (2), we will immediately obtain $d_1=d_2$. Then, assume $b=c$ and we must show $d_2=d_3$. Again, by substituting $b=c$ into (2) and (3), we directly obtain $d_2=d_3$. This proof indicates that if $a=b=c$ then $d_1=d_2=d_3$.

Next, suppose $d_1=d_2=d_3$ and we need to prove $a=b=c$. Since $d_1=d_2$, we will have

$$\frac{abc}{2}\int_0^\infty \frac{dy}{(y+a^2)^{3/2}\sqrt{(y+b^2)(y+c^2)}} = \frac{abc}{2}\int_0^\infty \frac{dy}{(y+b^2)^{3/2}\sqrt{(y+a^2)(y+c^2)}},$$

which implies $\frac{1}{y+a^2}=\frac{1}{y+b^2}$. After some derivations, we have $a^2=b^2$ which also means $a=\pm b$. Hence, it is clear that $a=b$ because $a,b>0$.

Now, let $d_2=d_3$. We must show $b=c$. By considering the formula of d_2 and d_3 from (2) and (3), we have

$$\frac{abc}{2}\int_0^\infty \frac{dy}{(y+b^2)^{3/2}\sqrt{(y+a^2)(y+c^2)}} = \frac{abc}{2}\int_0^\infty \frac{dy}{(y+c^2)^{3/2}\sqrt{(y+a^2)(y+b^2)}},$$

and further derivation will give $\frac{1}{y+b^2}=\frac{1}{y+c^2}$. Then, we obtain $b^2=c^2$ and since $b,c>0$, $b=c$. By combining all proofs, we have shown that if $d_1=d_2=d_3$ then $a=b=c$.

□

Corollary 6 Let a,b and c be the semi principal axes of ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and d_i is the depolarization factors for $i=1,2,3$.

i. $a=b$ if and only if $d_1=d_2$.
ii. $a=c$ if and only if $d_1=d_3$.
iii. $b=c$ if and only if $d_2=d_3$.

Proof We initially come out with this corollary in our previous studies and the proof is already given in [36].

□

In addition, by using Proposition 2 and Corollary 6, we can easily show that $d_1=d_2=d_3=\frac{1}{3}$ when $a=b=c$ without using (1), (2) and (3). Similarly, we can easily show that

$$d_3 = 1-2d_1 = 1-2d_2 \text{ when } a=b,$$
$$d_2 = 1-2d_1 = 1-2d_3 \text{ when } a=c,$$
$$d_1 = 1-2d_2 = 1-2d_3 \text{ when } b=c.$$

For spheroids with semi principal axes $a>b=c$ and $a<b=c$, the formula for d_1 can be further simplified, as shown by Stoner [5] and Milton [9]. We state their results in the next proposition. The proofs for Proposition 7 are already given by Stoner [5]. However, in this paper, we modified the technique to prove it for simplification.

Proposition 7 Let a,b and c be the semi principal axes of ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and d_1 is the depolarization factor for the ellipsoid in the x direction.

i. If $a>b=c$ then $d_1=\frac{1-\varepsilon^2}{\varepsilon^2}\left[\frac{1}{2\varepsilon}\ln\left(\frac{1+\varepsilon}{1-\varepsilon}\right)-1\right]$, where $\varepsilon=\sqrt{1-\left(\frac{b}{a}\right)^2}$.

ii. If $a<b=c$ then $d_1=\frac{1}{\varepsilon^2}\left[1-\frac{(1-\varepsilon^2)^{1/2}}{\varepsilon}\sin^{-1}\varepsilon\right]$, where $\varepsilon=\sqrt{1-\left(\frac{a}{b}\right)^2}$.

Proof For part (i), suppose $a>b=c$. Define $\varepsilon=\sqrt{1-\left(\frac{b}{a}\right)^2}$ where $0<\varepsilon<1$. We must show that $d_1=\frac{1-\varepsilon^2}{\varepsilon^2}\left[\frac{1}{2\varepsilon}\ln\left(\frac{1+\varepsilon}{1-\varepsilon}\right)-1\right]$. For convenience, we will introduce a symbol which is $\mu=\frac{b}{a}$. Thus, ε

can also be written as $\varepsilon = \sqrt{1-\mu^2}$. By using the substitution $z^2 = \frac{a^2 - b^2}{a^2 + y}$ given by Stoner [5] into (1), (1) can be reduced to

$$d_1 = \frac{\mu^2}{(1-\mu^2)^{3/2}} \int_0^{(1-\mu^2)^{1/2}} \frac{z^2}{1-z^2} dz. \qquad (16)$$

The integrand in (16) can be written as

$$\frac{z^2}{1-z^2} = \frac{1}{1-z^2} - 1,$$

so, the term d_1 in (16) can be expressed as

$$d_1 = \frac{\mu^2}{(1-\mu^2)^{3/2}} \int_0^{(1-\mu^2)^{1/2}} \frac{1}{1-z^2} - 1\, dz$$

and solved using integration involving inverse hyperbolic function and natural logarithm. This can be further simplified to

$$d_1 = \frac{1-\varepsilon^2}{\varepsilon^2}\left[\frac{1}{2\varepsilon}\ln\left(\frac{1+\varepsilon}{1-\varepsilon}\right) - 1\right]$$

which is defined for $0 < \varepsilon < 1$.

Proceed to prove part (ii), assume $a < b = c$ and define $\varepsilon = \sqrt{1-\left(\frac{a}{b}\right)^2}$ such that $0 < \varepsilon < 1$. We need to show $d_1 = \frac{1}{\varepsilon^2}\left[1 - \frac{(1-\varepsilon^2)^{1/2}}{\varepsilon}\sin^{-1}\varepsilon\right]$. Similarly, first we let $\eta = \frac{a}{b}$ so that $\varepsilon = \sqrt{1-\eta^2}$. Again, by using the substitution in Stoner [5], (1) can be expressed as

$$d_1 = \frac{\eta}{(1-\eta^2)^{3/2}} \int_0^{(1-\eta^2)^{1/2}} \frac{z^2}{(1-z^2)^{3/2}}\, dz. \qquad (17)$$

The integral expression (17) can be simplified by using the following relation

$$\int_0^{(1-\eta^2)^{1/2}} \frac{d}{dz}\frac{z}{(1-z^2)^{1/2}}\, dz = \int_0^{(1-\eta^2)^{1/2}} \left[\frac{z^2}{(1-z^2)^{3/2}} + \frac{1}{(1-z^2)^{1/2}}\right] dz.$$

and became

$$d_1 = \frac{\eta}{(1-\eta^2)^{3/2}}\left[\int_0^{(1-\eta^2)^{1/2}} \frac{d}{dz}\frac{z}{(1-z^2)^{1/2}}\, dz - \int_0^{(1-\eta^2)^{1/2}} \frac{1}{(1-z^2)^{1/2}}\, dz\right].$$

This can be further reduced when we consider the standard integral

$$\int \frac{1}{(1-z^2)^{1/2}}\, dz = \sin^{-1} z.$$

Thus, (17) can be expressed as

$$d_1 = \frac{1}{\varepsilon^2}\left[1 - \frac{(1-\varepsilon^2)^{1/2}}{\varepsilon}\sin^{-1}\varepsilon\right]$$

which is defined for $0 < \varepsilon < 1$.

□

DISCUSSION AND CONCLUSION

In this paper, the depolarization factors for ellipsoid and its properties were revised. A few other useful properties of the depolarization factors were also determined with the hope to apply them in our future researches. Firstly, we showed that the values of the depolarization factors should always be positive. Next, the relation indicating that the summation of the depolarization factors must equal to one was explained. Specifically, we then proved that the values for each depolarization factor must actually between zero and one. In this study, it was also presented that the semi principal axes of an ellipsoid can be categorized according to its depolarization factors. Reversely, it was shown that the value of the depolarization factors also depended on the semi principal axes of the ellipsoid. Finally, for two types of ellipsoid, we revised that their depolarization factors can be determined without using integrals of the general formula for depolarization factors.

ACKNOWLEDGEMENT

The authors would like to thank Ministry of Higher Education Malaysia (MOHE) for providing the MyBrainSc scholarship and Research Management Centre-UTM for financially supporting the study through the vote number 02K98. They are also very grateful to Che Lokman Jaafar from Universiti Teknologi Malaysia (UTM) for his useful suggestions during this research.

REFERENCES

[1] Chen, D. X., Pardo, E., and Sanchez, A. 2005. Demagnetizing factors for rectangular prisms. *IEEE Transactions on Magnetics*, 41(6): 2077-2088.
[2] Maxwell, J. C. 1954. *A treatise on electricity and magnetism*. Vol 2. New York: Dover Publications.
[3] Thomson, W. and Tait, P. G. 1895. *Treatise on natural philosophy*. London: Cambridge University Press.
[4] Osborn, J. 1945. Demagnetizing factors of the general ellipsoid. *Physical Review*, 67(11-12): 351-357.
[5] Stoner, E. C. 1945. The demagnetizing factors for ellipsoids. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 36(263): 803-821.
[6] Aharoni, A. 1998. Demagnetizing factors for rectangular ferromagnetic prisms. *Journal of Applied Physics*, 83(6): 3432-3434.
[7] Beleggia, M., De Graef, M., Millev, Y. T., Goode, D. A., and Rowlands, G. 2005. Demagnetization factors for elliptic cylinders. *Journal of Physics D: Applied Physics*, 38(18): 3333.
[8] Beleggia, M., De Graef, M. and Millev, Y. T. 2006. The equivalent ellipsoid of a magnetized body. *Journal of Physics D: Applied Physics*, 39(5): 891.
[9] Milton, G. 2002. *Theory of composites*. Cambridge, UK: Cambridge University Press.
[10] Ammari, H. and Kang, H. 2007. Polarization and moment tensors: With applications to inverse problems and effective medium theory. *Applied Mathematical Sciences Series 162*. New York: Springer-Verlag.
[11] Marsh, L. A., Ktistis, C., Jarvi, A., Armitage, D. W. and Peyton, A. J. 2013. Three-dimensional object location and inversion of the magnetic polarizability tensor at a single frequency using a walk-through metal detector. *Measurement Science and Technology*, 24(4): 1-13.
[12] Marsh, L. A., Ktistis, C., Jarvi, A., Armitage, D. W. and Peyton, A. J. 2014. Determination of the magnetic polarizability tensor and three dimensional object location for multiple objects using a walk-through metal detector. *Measurement Science and Technology*, 25(5): 1-12.
[13] Dekdouk, B., Marsh, L. A., Armitage, D. W. and Peyton, A. J. 2013. Estimating magnetic polarizability tensor of buried metallic targets for landmine clearance. In: Sabath F., Mokole E. (eds) *Ultra-Wideband, Short-Pulse Electromagnetics 10* (pp. 425-432). New York, NY: Springer.
[14] Khairuddin, T. K. A. and Lionheart, W. R. B. 2014. Does electro-sensing fish use the first order polarization tensor for object characterization? Object discrimination test. *Sains Malaysiana*, 43(11): 1775-1779.
[15] Ahmad Khairuddin, T. K. and Lionheart, W. R. B. 2016. Polarization tensor: Between biology and engineering. *Malaysian Journal of Mathematical Sciences*, 10(S): 179-191.

[16] Ahmad Khairuddin, T. K. and Lionheart, W. R. B. 2016. Characterization of objects by electrosensing fish based on the first order polarization tensor. *Bioinspiration and Biomimetics*, 11(5): 1-8.

[17] Bazeille, S., Lebastard, V., Lanneau, S. and Boyer, F. 2017. Model based object localization and shape estimation using electric sense on underwater robots. *IFAC-PapersOnLine*, 50(1): 5047-5054.

[18] Polya, G. 1947. A minimum problem about the motion of a solid through a fluid. *Proceedings of the National Academy of Sciences*, 33(7): 218-221.

[19] Schiffer, M. and Szego, G. 1949. Virtual mass and polarization. *Transactions of the American Mathematical Society*, 67(1): 130-205.

[20] Ammari, H., Boulier, T. and Garnier, J. 2013. Modeling active electrolocation in weakly electric fish. *SIAM Journal on Imaging Sciences*, 6(1): 285-321.

[21] Ammari, H., Boulier, T., Garnier, J. and Wang, H. 2014. Shape recognition and classification in electro-sensing. *Proceedings of the National Academy of Sciences*, 111(32): 11652-11657.

[22] Ammari, H., Boulier, T., Garnier, J. and Wang, H. 2017. Mathematical modelling of the electric sense of fish: The role of multifrequency measurements and movement. *Bioinspiration and Biomimetics*, 12(2): 1-16.

[23] Ammari, H., Vogelius, M. S. and Volkov, D. 2001. Asymptotic formula for perturbations in the electromagnetic fields due to the presence of inhomogeneities of small diameter II. The full Maxwell equations. *Journal de Mathématiques Pures et Appliquées*, 80(8): 769-814.

[24] Ammari, H., Chen, J., Chen, Z., Garnier, J. and Volkov, D. 2014. Target detection and characterization from electromagnetic induction data. *Journal de Mathématiques Pures et Appliquées*, 101(1): 54-75.

[25] Ammari, H., Chen, J., Chen, Z., Volkov, D. and Wang, H. 2015. Detection and classification from electromagnetic induction data. *Journal of Computational Physics*, 301: 201-217.

[26] Ledger, P. D. and Lionheart, W. R. B. 2015. Characterizing the shape and material properties of hidden targets from magnetic induction data. *IMA Journal of Applied Mathematics*, 80(6): 1776–1798.

[27] Ledger, P. D. and Lionheart,W. R. B. 2016. Understanding the magnetic polarizibility tensor. *IEEE Transactions on Magnetics*, 52(5): 1-16.

[28] Ahmad Khairuddin, T. K. and Lionheart, W. R. B. 2013. Some properties of the first order polarization tensor for 3-D domains. *Matematika*, 29(1): 1-18.

[29] Ahmad Khairuddin, T. K. and Lionheart, W. R. B. 2013. Computing the first order polarization tensor: Welcome BEM++!. *Menemui Matematik*, 35(2): 15-20.

[30] Ahmad Khairuddin, T. K. and Lionheart, W. R. B. 2015. Numerical comparisons for the approximated first order polarization tensor for ellipsoids. *Applied Mathematics and Computational Intelligence*, 4(1): 341-354.

[31] Khairuddin, T. K. A., Ledger, P. D. and Lionheart, W. R. B. 2015. Investigating the polarization tensor to describe and identify metallic objects. *Proceedings of the World Congress on Engineering*, 1: 122-127.

[32] Ahmad Khairuddin, T. K. and Lionheart, W. R. B. 2013. Fitting ellipsoids to objects by the first order polarization tensor. *Malaya Journal of Matematik*, 4(1): 44-53.

[33] Ahmad Khairuddin, T. K., Mohamad Yunos, N., Aziz, Z. A., Ahmad, T. and Lionheart, W. R. B. 2017. Classification of materials for conducting spheroids based on the first order polarization tensor. *Journal of Physics: Conference Series*, 890(1): 1-6.

[34] Mohamad Yunos, N. and Ahmad Khairuddin, T. K. 2017. Adapting depolarization factors in the first order polarization tensor for spheroid. *Final Year Project Proceeding (Department of Mathematical Sciences, UTM JB)*, 2: 383-390.

[35] Mohamad Yunos, N. and Ahmad Khairuddin, T. K. 2017. Describing rotations of the spheroids based on the first order polarization tensor. *eProceeding Chemistry*, 2: 297-302.

[36] Mohamad Yunos, N., Ahmad Khairuddin, T. K. and Lionheart, W. R. B. 2017. Identification of a spheroid based on the first order polarization tensor. *Journal of Science and Technology*, 9(3): 154-159.