

Inequalities on weighted classical pythagorean means, Tracy-Singh products, and Khatri-Rao products for hermitian operators

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Abstract

We establish a number of operator inequalities between three kinds of means, namely, weighted arithmetic/harmonic/geometric means, and two kinds of operator products, namely, Tracy-Singh products and Khatri-Rao products. In this study, we have validated the data under certain assumptions relying on (opposite) synchronization, comparability, and spectra of operators. The tensor product of operators, and Tracy-Singh/Khatri-Rao products of matrices as special cases are presented.

Keywords: weighted arithmetic mean, weighted geometric mean, weighted harmonic mean, Tracy-Singh product, Khatri-Rao product

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INTRODUCTION

Throughout this paper, H and K denote the complex separable Hilbert spaces. When X and Y are Hilbert spaces, let $B(X, Y)$ be the algebra of all bounded linear operators from X into Y , and abbreviate $B(X, X)$ to $B(X)$. Denoted by $B(H)^+$ the cone of positive operators on H . For any Hermitian operators A and B on H , the partial order $A \leq B$ indicates that $A - B \in B(H)^+$. Two Hermitian operators A and B are comparable if $A \leq B$ or $B \leq A$. The expressions $A > 0$ and $A \in B(H)^{++}$ have the same meaning so that A is both positive and invertible.

This paper focuses on the three classical Pythagorean means, namely, the arithmetic mean, the geometric mean, and the harmonic mean. Over the years, theory of these kinds of means for matrices and operators are significantly developed, see e.g. [1-3] and references therein. Recall that for any $t \in [0, 1]$, the t -weighted arithmetic mean of $A, B \in B(H)$ is defined by

$$A^#_t B = (1-t)A + tB.$$

The t -weighted harmonic mean of $A, B \in B(H)^{++}$ is defined by

$$A!_t B = \left[(1-t)A^{-1} + tB^{-1} \right]^{-1}.$$

In general, for any $A, B \in B(H)^+$, we define

$$A!_t B = \lim_{\varepsilon \rightarrow 0^+} (A + \varepsilon I)!_t (B + \varepsilon I). \tag{1}$$

Here, the limit is taken in the strong-operator topology. The t -weighted geometric mean of two operators $A, B \in B(H)^+$ is defined by

$$A \#_t B = A^{1/2} \left(A^{-1/2} B A^{-1/2} \right)^t A^{1/2}.$$

For arbitrary positive operators, we define their weighted geometric mean by using the continuity argument as that for (1). In brief, we write $A^#_t B, A!_t B$ and $A \#_t B$ for $A^#_{1/2} B, A!_{1/2} B$ and $A \#_{1/2} B$, respectively.

Weighted classical Pythagorean means have the following remarkable properties where for any $A, B \in B(H)^+, X \in B(H)$, and $t \in [0, 1]$, we have

$$A \sigma_t B = B \sigma_{1-t} A, \tag{2}$$

$$X^* (A \sigma_t B) X \leq (X^* A X) \sigma_t (X^* B X), \tag{3}$$

It is well-known that for any $A_1, B_1, A_2, B_2 \in B(H)^+$, we have

$$(A_1 \# B_1) \otimes (A_2 \# B_2) = (A_1 \otimes A_2) \# (B_1 \otimes B_2).$$

Here, \otimes denotes the tensor product. Recently, the theory of tensor product for operators is extended to that of Tracy-Singh product and Khatri-Rao product for operators, see e.g. [4-7]. The identity shown was generalized to that for weighted geometric means and Tracy-Singh products as follows:

Proposition 1 ([8]). For any $A_1, B_1, A_2, B_2 \in B(H)^+$ and $t \in [0, 1]$, we have

$$(A_1 \#_t B_1) \otimes (A_2 \#_t B_2) = (A_1 \otimes A_2) \#_t (B_1 \otimes B_2). \tag{4}$$

In this paper, we establish further inequalities between three kinds of weighted Pythagorean means, namely, weighted arithmetic/harmonic/geometric means, and two kinds of operator products, namely, Tracy-Singh product and Khatri-Rao product. Our results include tensor product of operators, and Tracy-Singh/Khatri-Rao products of matrices as special cases.

The outline for the rest of paper is as follows. In Section 2, we present the preliminary results on Tracy-Singh product and Khatri-Rao product of Hilbert space operators. Section 3 begins with introducing the (opposite) synchronization between two ordered pairs of Hermitian operators. Then, we establish operator inequalities involving Tracy-Singh products and weighted arithmetic/harmonic means under the assumptions that two pairs of operators are (opposite) synchronous. In Section 4, we prove certain operator inequalities concerning Khatri-Rao products and weighted arithmetic/harmonic/geometric means under suitable assumptions about synchronization, comparability, and spectra of operators.

PRELIMINARIES ON TRACY-SINGH PRODUCT AND KHATRI-RAO PRODUCT OF HILBERT SPACE OPERATORS

In order to define the Tracy-Singh product for Hilbert space operators, we have to fix the following orthogonal decompositions:

$$H = \bigoplus_{i=1}^m H_i, \quad K = \bigoplus_{k=1}^n K_k \tag{5}$$

where H_i and K_j are Hilbert spaces for i, j . Thus, each operator $A \in B(H)$ and $B \in B(K)$ can be uniquely represented as operator matrices

$$A = [A_{ij}]_{i,j=1}^{m,m} \quad \text{and} \quad B = [B_{kl}]_{k,l=1}^{n,n} \tag{6}$$

where $A_{ij} \in B(H_j, H_i)$ and $B_{kl} \in B(K_l, K_k)$ for each $i, j = 1, \dots, m$ and $k, l = 1, \dots, n$.

Definition 2. Let $A = [A_{ij}]_{i,j=1}^{m,m}$ and $B = [B_{kl}]_{k,l=1}^{n,n}$ be operator matrices in $B(H)$ and $B(K)$, respectively. We define the Tracy-Singh product of A and B to be the bounded linear operator from $\bigoplus_{i,k=1}^{m,n} H_i \otimes K_k$ into itself, represented by

$$A \circ B = \left[[A_{ij} \otimes B_{kl}]_{ij} \right]_{kl} \tag{7}$$

Lemma 3 ([4]). Let A_1, B_1, A_2, B_2 be compatible operators. Then

- (i) $(A_1 + B_1) \circ (A_2 + B_2) = (A_1 \circ A_2 + A_1 \circ B_2 + B_1 \circ A_2 + B_1 \circ B_2)$.
- (ii) $(\alpha A_1) \circ A_2 = \alpha (A_1 \circ A_2)$ and $A_1 \circ (\alpha A_2) = \alpha (A_1 \circ A_2)$ for any $\alpha \in \mathbb{C}$.
- (iii) $(A_1 \circ A_2) \circ (B_1 \circ B_2) = (A_1 \circ B_1) \circ (A_2 \circ B_2)$.
- (iv) If A_1 and A_2 are invertible, then $(A_1 \circ A_2)^{-1} = A_1^{-1} \circ A_2^{-1}$.
- (v) If $A_1 \circ A_2 \circ 0$ and $B_1 \circ B_2 \circ 0$, then $A_1 \circ B_1 \circ A_2 \circ B_2 \circ 0$.

To define the Khatri-Rao product of operators, we fix the decomposition (5) and assume that $m = n$.

Definition 4. Let $A = [A_{ij}]_{i,j=1}^{n,n} \in B(H)$ and $B = [B_{ij}]_{i,j=1}^{n,n} \in B(K)$. The Khatri-Rao product of A and B is defined to be the bounded linear operator from $\bigoplus_{i=1}^n H_i \otimes K_i$ into itself, represented by

$$A * B = [A_{ij} \otimes B_{ij}]_{i,j=1}^{n,n} \tag{8}$$

Lemma 5. ([5]). There exists a bounded linear operator Z such that $A * B = Z^n(A \circ B)Z$ for any $A \in B(H)$ and $B \in B(K)$.

The operator Z in Lemma 5 is called the selection operator associated with the ordered tuple (H, K) .

OPERATOR INEQUALITIES ON WEIGHTED CLASSICAL PYTHAGOREAN MEANS AND TRACY-SINGH PRODUCTS

In this section, we establish operator inequalities involving weighted arithmetic/harmonic means and Tracy-Singh products under the assumptions that two pairs of operators are (opposite) synchronous.

Definition 6. Two ordered pairs (A_1, A_2) and (B_1, B_2) of Hermitian operators are said to be synchronous if either

$$A_i \tilde{N} B_i \text{ for } i=1,2, \text{ or } A_i \tilde{O} B_i \text{ for } i=1,2.$$

The pairs (A_1, A_2) and (B_1, B_2) are said to be opposite synchronous if either

$$A_1 \tilde{N} B_1 \text{ and } A_2 \tilde{O} B_2, \text{ or } A_1 \tilde{O} B_1 \text{ and } A_2 \tilde{N} B_2.$$

Now, we establish operator inequalities involving weighted harmonic means and Tracy-Singh products.

Theorem 7. Let $A_1, B_1, A_2, B_2 \in B(H)^+$ and $t \in [0,1]$.

(i) If (A_1, A_2) and (B_1, B_2) are synchronous, then
$$(A_1 \sharp B_1) \circ (A_2 \sharp B_2) \tilde{O} (A_1 \circ A_2) \sharp (B_1 \circ B_2). \tag{10}$$

(ii) If (A_1, A_2) and (B_1, B_2) are opposite synchronous, then
$$(A_1 \sharp B_1) \circ (A_2 \sharp B_2) \tilde{N} (A_1 \circ A_2) \sharp (B_1 \circ B_2). \tag{11}$$

Proof. (i) First, suppose $A_1, B_1, A_2, B_2 > 0$. The case $A_1 \tilde{O} B_1, A_2 \tilde{O} B_2$ leads to $A_1^{-1} \tilde{N} B_1^{-1}$ and $A_2^{-1} \tilde{N} B_2^{-1}$. Thus, by Lemma 3 we have

$$(A_1^{-1} - B_1^{-1}) \circ (A_2^{-1} - B_2^{-1}) \tilde{O} 0. \tag{12}$$

The case $A_1 \tilde{N} B_1, A_2 \tilde{N} B_2$ also leads to the inequality (12). It follows from (12) that

$$\begin{aligned} & 0 \tilde{N} t(1-t)A_1^{-1} \circ A_2^{-1} + t(1-t)B_1^{-1} \circ B_2^{-1} - t(1-t)A_1^{-1} \circ B_2^{-1} \\ & \quad - t(1-t)B_1^{-1} \circ A_2^{-1} \\ & = [(1-t) - (1-t)^2]A_1^{-1} \circ A_2^{-1} + (t-t^2)B_1^{-1} \circ B_2^{-1} \\ & \quad - t(1-t)A_1^{-1} \circ B_2^{-1} - t(1-t)B_1^{-1} \circ A_2^{-1} \\ & = (1-t)A_1^{-1} \circ A_2^{-1} + tB_1^{-1} \circ B_2^{-1} \\ & \quad - [(1-t)A_1^{-1} + tB_1^{-1}] \circ [(1-t)A_2^{-1} + tB_2^{-1}]. \end{aligned}$$

Thus

$$\begin{aligned} & [(1-t)A_1^{-1} + tB_1^{-1}] \circ [(1-t)A_2^{-1} + tB_2^{-1}] \\ & \quad \tilde{N} (1-t)A_1^{-1} \circ A_2^{-1} + tB_1^{-1} \circ B_2^{-1}. \end{aligned}$$

Hence

$$\begin{aligned} & (A_1 \circ A_2) \sharp (B_1 \circ B_2) \\ & = \{(1-t)(A_1 \circ A_2)^{-1} + t(B_1 \circ B_2)^{-1}\}^{-1} \\ & = \{(1-t)A_1^{-1} \circ A_2^{-1} + tB_1^{-1} \circ B_2^{-1}\}^{-1} \\ & \tilde{N} \{[(1-t)A_1^{-1} + tB_1^{-1}] \circ [(1-t)A_2^{-1} + tB_2^{-1}]\}^{-1} \\ & = [(1-t)A_1^{-1} + tB_1^{-1}]^{-1} \circ [(1-t)A_2^{-1} + tB_2^{-1}]^{-1} \\ & = (A_1 \sharp B_1) \circ (A_2 \sharp B_2). \end{aligned}$$

For arbitrary $A_1, B_1, A_2, B_2 \circ 0$, perturb each of them with εI and then take limit as $\varepsilon \rightarrow 0^+$.

(ii) By continuity, we may assume that $A_1, B_1, A_2, B_2 > 0$. If (A_1, A_2) and (B_1, B_2) are opposite synchronous, then we get the reverse of (12). Hence, in this case, we get (11).

Corollary 8. If two positive operators A and B are comparable, then for any $t \in [0,1]$,

$$(A \sharp B) \circ (A \sharp B) \tilde{O} (A \circ A) \sharp (B \circ B), \tag{13}$$

$$(A \sharp B) \circ (B \sharp A) \tilde{N} (A \circ B) \sharp (B \circ A), \tag{14}$$

$$(A \sharp B) \circ (B^{-1} \sharp A^{-1}) \tilde{O} (A \circ B^{-1}) \sharp (B \circ A^{-1}), \tag{15}$$

$$(A \sharp B) \circ (A^{-1} \sharp B^{-1}) \tilde{N} (A \circ A^{-1}) \sharp (B \circ B^{-1}). \tag{16}$$

Here, in (15) and (16), we assume further that A and B are invertible.

Proof. First, suppose $A \tilde{N} B$. Then the pairs (A, A) and (B, B) are synchronous. By Theorem 7, we get (13). Since (A, B) and (B, A) are opposite synchronous, Theorem 7 yields the inequality (14). The synchronization between (A, B^{-1}) and (B, A^{-1}) implies the inequality (15). The opposite synchronization between (A, A^{-1}) and (B, B^{-1}) implies the inequality (16). The case $A \tilde{O} B$ can be similarly treated.

Next, we discuss relations between weighted arithmetic means and Tracy-Singh products in terms of inequalities.

Theorem 9. Let $A_1, B_1, A_2, B_2 \in B(H)$ be Hermitian operators and $t \in [0,1]$.

(i) If (A_1, A_2) and (B_1, B_2) are synchronous, then

$$(A_1 \#_t B_1) (A_2 \#_t B_2) \tilde{N} (A_1 A_2) \#_t (B_1 B_2). \tag{17}$$

(ii) If (A_1, A_2) and (B_1, B_2) are opposite synchronous, the

$$(A_1 \#_t B_1) (A_2 \#_t B_2) \tilde{O} (A_1 A_2) \#_t (B_1 B_2). \tag{18}$$

Proof. (i) Since (A_1, A_2) and (B_1, B_2) are synchronous, by Lemma 3 we have $(A_1 - B_1) (A_2 - B_2) \tilde{O} 0$. By using Lemma 3 again, we obtain

$$\begin{aligned} & 0 \tilde{N} t(1-t)[(A_1 - B_1) (A_2 - B_2)] \\ &= t(1-t)[(A_1 - A_2 - B_1 + B_2) (A_2 + B_1 - B_2)] \\ &= [(1-t)(A_1 - A_2) + t(B_1 - B_2)] \\ &\quad - [(1-t)A_1 + tB_1] [(1-t)A_2 + tB_2] \\ &= [(A_1 - A_2) \#_t (B_1 - B_2)] - [(A_1 \#_t B_1) (A_2 \#_t B_2)]. \end{aligned}$$

Thus $(A_1 \#_t B_1) (A_2 \#_t B_2) \tilde{N} (A_1 A_2) \#_t (B_1 B_2)$.

(ii) For the opposite synchronous case, we have $(A_1 - B_1) (A_2 - B_2) \tilde{N} 0$ and hence the inequality (18) holds.

Corollary 10. If two Hermitian operators A and B are comparable, then for any $t \in [0,1]$,

$$(A \#_t B) (A \#_t B) \tilde{N} (A A) \#_t (B B), \tag{19}$$

$$(A \#_t B) (B \#_t A) \tilde{O} (A B) \#_t (B A), \tag{20}$$

$$(A \#_t B) (B^{-1} \#_t A^{-1}) \tilde{N} (A B^{-1}) \#_t (B A^{-1}), \tag{21}$$

$$(A \#_t B) (A^{-1} \#_t B^{-1}) \tilde{O} (A A^{-1}) \#_t (B B^{-1}). \tag{22}$$

Here, in (21) and (22), we assume further that A and B are invertible.

Proof. The proof is similar to that of Corollary 8.

OPERATOR INEQUALITIES ON WEIGHTED CLASSICAL PYTHAGOREAN MEANS AND KHATRI-RAO PRODUCTS

In this section, we present a number of operator inequalities involving Khatri-Rao products and weighted arithmetic/harmonic/geometric means under suitable assumptions about synchronization, comparability, and spectra of operators.

Corollary 11. Let $A, B \in B(H)^+$ be comparable operators. If $A \# B = B \# A$, then for any $t \in [0,1]$,

$$A \# B \tilde{O} (A \#_t B) \# (A \#_{1-t} B).$$

Proof. By making use of Lemma 5, property (3), Corollary 10 and property (2), we obtain

$$\begin{aligned} A \# B &= (A \# B) \#_t (B \# A) \\ &= [Z^*(A \# B)Z] \#_t [Z^*(B \# A)Z] \\ &\tilde{O} Z^*[(A \# B) \#_t (B \# A)]Z \\ &\tilde{O} Z^*[(A \#_t B) (B \#_t A)]Z \\ &= (A \#_t B) \# (B \#_t A) = (A \#_t B) \# (A \#_{1-t} B). \end{aligned}$$

Theorem 12. Let $A, B \in B(H)^+$ and $t \in (0,1)$. If $A \# B = B \# A$, then $A \# B \tilde{O} (A \#_t B) \# (A \#_{1-t} B)$.

Proof. It follows Lemma 5, property (3), Proposition 1, and property (2) that

$$\begin{aligned} A \# B &= (A \# B) \#_t (B \# A) \\ &= [Z^*(A \# B)Z] \#_t [Z^*(B \# A)Z] \\ &\tilde{O} Z^*[(A \# B) \#_t (B \# A)]Z \\ &= Z^*[(A \#_t B) (B \#_t A)]Z \\ &= (A \#_t B) \# (B \#_t A) \\ &= (A \#_t B) \# (A \#_{1-t} B). \end{aligned}$$

Corollary 13. For any $X, Y \in B(H)^+$ such that Y is invertible, we have

$$Y \# (XY^{-1}X) \tilde{O} X \# X.$$

Proof. Note that for any $A, B \in B(H)^+$ such that A is invertible, we have $A \# (BA^{-1}B) = B$. It follows this fact and Theorem 12 with $t = 1/2$ that

$$Y \# (XY^{-1}X) \tilde{O} (Y \# XY^{-1}X) \# (Y \# XY^{-1}X) = X \# X.$$

Theorem 14. Let $A, B \in B(H)^+$ and $t \in (0,1)$. If $A \# B = B \# A$, then

$$A \# B \tilde{O} (A \#_t B) \# (A \#_{1-t} B).$$

Proof. Recall that the Khatri-Rao product is continuous with respect to the operator norm (see [9]). By continuity of the Khatri-Rao product, the t -weighted arithmetic mean and the t -weighted harmonic mean, we may assume that A and B are invertible.

$$(X^{-1} + Y^{-1})^{-1} = X - X(X + Y)^{-1}X.$$

It follows this fact and Corollary 13 that

$$\begin{aligned} & (A \#_t B) \# (A \#_{1-t} B) \\ &= [(1-t)A + tB] \# [tA^{-1} + (1-t)B^{-1}]^{-1} \\ &= [(1-t)A + tB] \# t^{-1} \{ A - (1-t)A[(1-t)A + tB]^{-1}A \} \\ &= [(1-t)A + tB] \# t^{-1}A \\ &\quad - [(1-t)A + tB] \# \{ t^{-1}(1-t)A[(1-t)A + tB]^{-1}A \} \\ &= t^{-1}(1-t)(A \# A) + B \# A \\ &\quad - t^{-1}(1-t)[(1-t)A + tB] \# \{ A[(1-t)A + tB]^{-1}A \} \\ &\tilde{N} t^{-1}(1-t)(A \# A) + A \# B - t^{-1}(1-t)(A \# A) \\ &= A \# B. \end{aligned}$$

In [8], it was shown that for any A_1, B_1, A_2, B_2 in $B(H)^+$ and $t \in [0,1]$,

$$(A_1 \#_t B_1) \# (A_2 \#_t B_2) \tilde{N} (A_1 \# A_2) \#_t (B_1 \# B_2). \tag{23}$$

Now, we give a reverse inequality of (23). Recall that a linear map Φ between two operator algebras is said to be positive if it preserves positive operators; Φ is said to be unital if it preserves the identity operator.

Lemma 15 ([10]). Let $A, B \in B(H)^+$ be such that $m_1 I \tilde{N} A \tilde{N} M_1 I$ and $m_2 I \tilde{N} B \tilde{N} M_2 I$ where m_1, m_2, M_1, M_2 are positive constants. Denote $m = m_2 / M_1$ and $M = M_2 / m_1$. Then, for any positive linear map $\Phi : B(H) \rightarrow B(H)$ and $t \in (0,1)$, we have

$$\Phi(A \#_t B) \tilde{O} \lambda [\Phi(A) \#_t \Phi(B)] \tag{24}$$

where

$$\lambda = \frac{Mm' - mM'}{(1-t)(M-m)} \cdot \left(\frac{1-t}{t} \cdot \frac{M' - m'}{Mm' - mM'} \right)^t \tag{25}$$

Theorem 16. Let $A_i, B_i, A_2, B_2 \in B(H)^{++}$ be such that

$$0 < m_1 I \tilde{N} A_1 \quad A_2 \tilde{N} M_1 I \quad \text{and} \quad 0 < m_2 I \tilde{N} B_1 \quad B_2 \tilde{N} M_2 I.$$

Let $m = m_2 / M_1$ and $M = M_2 / m_1$. Then for any $0 < t < 1$, we have

$$(A_1 \#_t B_1) * (A_2 \#_t B_2) \tilde{O} \lambda [(A_1 * A_2) \#_t (B_1 * B_2)] \tag{26}$$

where λ is given by (25).

Proof. Consider a map $\Phi: T \rightarrow Z^*TZ$ where Z is the selection operator described in Lemma 5, associated with the ordered tuple (H, H) . Then, Φ is a unital positive linear map. It follows Lemma 5, Proposition 1, and Lemma 15 that

$$\begin{aligned} (A_1 \#_t B_1) * (A_2 \#_t B_2) &= Z^* [(A_1 \#_t B_1)) (A_2 \#_t B_2)] Z \\ &= Z^* [(A_1) (A_2) \#_t (B_1) (B_2)] Z \\ \tilde{O} \lambda \{ [Z^* (A_1) (A_2) Z] \#_t [Z^* (B_1) (B_2) Z] \} \\ &= \lambda [(A_1 * A_2) \#_t (B_1 * B_2)]. \end{aligned}$$

Corollary 17. Let $A, B \in B(H)^{++}$ and $t \in (0, 1)$. Let m_1, m_2, M_1, M_2 be positive constants, denote $m = m_2 / M_1$ and $M = M_2 / m_1$, and define λ as in (25).

(i) If $m_1 I \tilde{N} A) A \tilde{N} M_1 I$ and $m_2 I \tilde{N} B) B \tilde{N} M_2 I$, then

$$(A \#_t B) * (A \#_t B) \tilde{O} \lambda [(A * A) \#_t (B * B)].$$

(ii) If $m_1 I \tilde{N} A) B \tilde{N} M_1 I$ and $m_2 I \tilde{N} B) A \tilde{N} M_2 I$, then

$$(A \#_t B) * (A \#_{1-t} B) \tilde{O} \lambda [(A * B) \#_t (B * A)].$$

Proof The assertion (i) follows from Theorem 16 by setting $A_1 = A_2 := A$ and $B_1 = B_2 := B$. To prove (ii), set $A_1 = B_2 := A$ and $B_1 = A_2 := B$ in the same theorem and property (2).

Recall the following relation between a positive linear map and the harmonic mean of operators:

Lemma 18 ([10]). Let $A, B \in B(H)^{++}$ be such that $0 < m_1 I \tilde{N} A \tilde{N} M_1 I$ and $0 < m_2 I \tilde{N} B \tilde{N} M_2 I$. If $\Phi: B(H) \rightarrow B(H)$ is a positive linear map, then

$$\Phi(A!B) \tilde{O} \frac{(\sqrt{M_1 m_1} + \sqrt{M_2 m_2})^2}{(M_1 + m_2)(m_1 + M_2)} [\Phi(A)! \Phi(B)].$$

Theorem 19. Let $A_i, B_i, A_2, B_2 \in B(H)^{++}$ and $t \in [0, 1]$.

(i) Suppose $0 < m_1 I \tilde{N} A_1) A_2 \tilde{N} M_1 I$ and $0 < m_2 I \tilde{N} B_1) B_2 \tilde{N} M_2 I$. If

$$(A_1, A_2) \text{ and } (B_1, B_2) \text{ are synchronous, then} \tag{27}$$

$$(A_1!_t B_1) * (A_2!_t B_2) \tilde{O} k^2 [(A_1 * A_2)!_t (B_1 * B_2)]$$

where

$$k = \frac{(m_1 \# M_1)_{1-t} (m_2 \# M_2)}{(m_1)_{1-t} M_2 \# (M_1)_{1-t} m_2} \tag{28}$$

(ii) If (A_1, A_2) and (B_1, B_2) are opposite synchronous, then

$$(A_1!_t B_1) * (A_2!_t B_2) \tilde{N} (A_1 * A_2)!_t (B_1 * B_2). \tag{29}$$

Proof. (i) Assume that (A_1, A_2) and (B_1, B_2) are synchronous. The cases $t = 0$ and $t = 1$ are trivial. Now, $t \in (0, 1)$. By using Lemmas 5 and 18, and Theorem 12, we have

$$\begin{aligned} (A_1!_t B_1) * (A_2!_t B_2) \\ = Z^* [(A_1!_t B_1)) (A_2!_t B_2)] Z \end{aligned}$$

$$\begin{aligned} &\tilde{O} Z^* [(A_1) (A_2)!_t (B_1) (B_2)] Z \\ &= Z^* \left\{ \left[\frac{1}{1-t} (A_1) (A_2) \right]!_t \left[\frac{1}{t} (B_1) (B_2) \right] \right\} Z \\ &\tilde{O} \frac{\left(\frac{\sqrt{M_1 m_1}}{1-t} + \frac{\sqrt{M_2 m_2}}{t} \right)^2}{\left(\frac{M_1}{1-t} + \frac{m_2}{t} \right) \left(\frac{m_1}{1-t} + \frac{M_2}{t} \right)} \\ &\quad \times \left\{ \left[\frac{1}{1-t} Z^* (A_1) (A_2) Z \right]!_t \left[\frac{1}{t} Z^* (B_1) (B_2) Z \right] \right\} \\ &= \left(\frac{(m_1 \# M_1)_{1-t} (m_2 \# M_2)}{(M_1)_{1-t} m_2 \# (m_1)_{1-t} M_2} \right)^2 [(A_1 * A_2)!_t (B_1 * B_2)]. \end{aligned}$$

Assume that (A_1, A_2) and (B_1, B_2) are opposite synchronous. By applying Lemma 5, Proposition 7 and property (3), we obtain that for any $t \in [0, 1]$,

$$\begin{aligned} (A_1!_t B_1) * (A_2!_t B_2) &= Z^* [(A_1!_t B_1)) (A_2!_t B_2)] Z \\ &\tilde{N} Z^* [(A_1) (A_2)!_t (B_1) (B_2)] Z \end{aligned}$$

$$\begin{aligned} &\tilde{N} [Z^* (A_1) (A_2) Z]!_t [Z^* (B_1) (B_2) Z] \\ &\tilde{N} (A_1 * A_2)!_t (B_1 * B_2). \end{aligned}$$

Corollary 20. Let $A, B \in B(H)^{++}$ and $t \in [0, 1]$. Assume that A and B are comparable. Then

$$(A!_t B) * (B!_t A) \tilde{N} (A * B)!_t (B * A), \tag{30}$$

$$(A!_t B) * (A^{-1}!_t B^{-1}) \tilde{N} (A * A^{-1})!_t (B * B^{-1}). \tag{31}$$

Moreover, if $0 < \sqrt{m_1} I \tilde{N} A \tilde{N} \sqrt{M_1} I$ and $0 < \sqrt{m_2} I \tilde{N} B \tilde{N} \sqrt{M_2} I$, then

$$(A!_t B) * (A!_t B) \tilde{O} k^2 [(A * A)!_t (B * B)]. \tag{32}$$

Here, the constant k is given by (28).

Proof. The results in this corollary are consequences of Theorem 4.9. The opposite synchronization between (A, B) and (B, A) leads to (30).

The opposite synchronization between (A, A^{-1}) and (B, B^{-1}) implies

$$(31). \text{ If } 0 < \sqrt{m_1} I \tilde{N} A \tilde{N} \sqrt{M_1} I \text{ and } 0 < \sqrt{m_2} I \tilde{N} B \tilde{N} \sqrt{M_2} I,$$

then by Lemma 3, we have $m_1 I \tilde{N} A) A \tilde{N} M_1 I$ and $m_2 I \tilde{N} B) B \tilde{N} M_2 I$, and the inequality (32) follows.

Theorem 21. Let $A_i, B_i, A_2, B_2 \in B(H)$ be Hermitian and $t \in [0, 1]$.

(i) If (A_1, A_2) and (B_1, B_2) are synchronous, then

$$(A_1)_{!_t B_1} * (A_2)_{!_t B_2} \tilde{N} (A_1 * A_2)_{!_t (B_1 * B_2)}. \tag{33}$$

(ii) If (A_1, A_2) and (B_1, B_2) are opposite asynchronous, the

$$(A_1)_{!_t B_1} * (A_2)_{!_t B_2} \tilde{O} (A_1 * A_2)_{!_t (B_1 * B_2)}. \tag{34}$$

Proof. (i) By using Lemma 5 and Theorem 9, we obtain

$$\begin{aligned} (A_1)_{!_t B_1} * (A_2)_{!_t B_2} &= Z^* [(A_1)_{!_t B_1}) (A_2)_{!_t B_2}] Z \\ &\tilde{N} Z^* [(A_1) (A_2)_{!_t (B_1) (B_2)}] Z \\ &= (1-t) Z^* (A_1) (A_2) Z + t Z^* (B_1) (B_2) Z \\ &= (1-t) (A_1 * A_2) + t (B_1 * B_2) \\ &= (A_1 * A_2)_{!_t (B_1 * B_2)}. \end{aligned}$$

(ii) The proof is similar to that of (i).

Corollary 22. If two Hermitian operators A and B are comparable, then

$$(A \text{ } _i B)^* (A \text{ } _i B) \tilde{N} (A^* A) \text{ } _i (B^* B), \tag{4.13}$$

$$(A \text{ } _i B)^* (B \text{ } _i A) \tilde{O} (A^* B) \text{ } _i (B^* A), \tag{4.14}$$

$$(A \text{ } _i B)^* (B^{-1} \text{ } _i A^{-1}) \tilde{N} (A^* B^{-1}) \text{ } _i (B^* A^{-1}), \tag{4.15}$$

$$(A \text{ } _i B)^* (A^{-1} \text{ } _i B^{-1}) \tilde{O} (A^* A^{-1}) \text{ } _i (B^* B^{-1}). \tag{4.16}$$

Here, in (37) and (38), we assume further that A and B are invertible.

Proof. The proof is similar to that of Corollary 8.

CONCLUSIONS

We provide a number of operator inequalities between three classical Pythagorean means and two kinds of operator products, namely, Tracy-Singh products and Khatri-Rao products. Each inequality is valid under certain assumptions relying on (opposite) synchronization, comparability, and spectra of operators. Our results include tensor product of operators, and Tracy-Singh and Khatri-Rao products of matrices as special cases.

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