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RESEARCH ARTICLE

The independence polynomial of inverse commuting graph of dihedral groups

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Abstract

Set of vertices not joined by an edge in a graph is called the independent set of the graph. The independence polynomial of a graph is a polynomial whose coefficient is the number of independent sets in the graph. In this research, we introduce and investigate the inverse commuting graph of dihedral groups (D_{2n}) denoted by G^{IC}. It is a graph whose vertex set consists of the non-central elements of the group and for distinct $x, y \in D_{2n}$, x and y are adjacent if and only if $xy = yx = 1$, where 1 is the identity element. The independence polynomials of the inverse commuting graph for dihedral groups are also computed. A formula for obtaining such polynomials without getting the independent sets is also found, which was used to compute for dihedral groups of order 18 up to 32.

Keywords: Dihedral group, inverse commuting graph, independent sets, independence polynomial

INTRODUCTION

A graph G is a pair of sets $G = (V, E)$, where V is a finite set of elements called vertices, E which is a set of unordered pairs of distinct elements of V called edges. The set V is called vertex set and E edge set [1]. This paper deals with finite, simple, and complete graphs. Finite graph has finite vertices and edges while simple graph is a graph in which two vertices are joined by at most a single edge. A complete graph on n vertices denoted by K_n is a graph in which each pair of distinct vertices is joined by an edge. There is a null graph which has 0 vertices and no edges denoted as $E_0 := \emptyset$ [2]. The paper also focuses on finite non-abelian groups. Let D_{2n} represent dihedral group, where

$$D_{2n} =< a, b : a^n = b^2 = 1, bab^{-1} = a^{-1} >.$$

In 1994, Hoede and Li [3] came up with the independence polynomial of graphs, which involves getting independent sets and independence number of the graphs.

There were researches on the independence polynomials of graphs for the dihedral groups; like the conjugate graphs, non-commuting graphs, and n-th central graphs [4,5] but there was none on the commuting graphs for the dihedral groups. Although the complement of the non-commuting graph of dihedral groups gives the commuting graph for the groups; the independence polynomial of non-commuting graph is in no way related to that of commuting graph. This is so since reflections commute for even n while for odd n there are no independent sets for the commuting graph; and without the independent sets, we cannot find independence polynomial using Hoede and Li's concept.

This means that one can only find the independence polynomial of commuting graphs of dihedral group with even n using Hoede and Li's idea.

This motivates the authors to investigate on the inverse commuting graphs of dihedral groups. The paper is organized as follows: Section 2 gives some preliminaries and methodology used in the work; Section 3 is presents on how the inverse commuting elements are obtained and used as vertices in constructing the inverse commuting graph; Section 4 computes the independence polynomials of the graphs; and Section 5 shows how to compute the independence polynomials of the graphs using a formula.

PRELIMINARIES

We look at some concepts that will aid in getting the independence polynomials of the graphs.

Definition 2.1 [2] Independent Set
This is the subset of the vertex set of the graph such that no two vertices in the subset are adjacent in the graph.

Definition 2.2 [2] Independence Number
The independence number of a graph is the number of elements in the independent sets with the highest cardinality, which we denote by *n(G)*.

Definition 2.3 [3] Independence Polynomial
Independence polynomial of a graph G is the polynomial whose co-efficient C_k on x^k is the number of independent sets of order k in G. It is given by; $I(G; x) = \sum_{k=0}^{n(G)} C_k x^k$

Definition 2.4 [3] Independence Polynomial for Null Graph
The independence polynomial of a null graph is given by $I(\emptyset; x) = 1$.

Definition 2.5 [6] Central Elements
This is the set of elements in a group that commute with every other element in the group.

The work begins by studying the Caley's table for the dihedral groups under consideration in order to see how to generate the inverse commuting elements which will then serve as the vertex set for the graphs.

RESULTS

Inverse commuting graph of dihedral groups

In this section, we constructed the inverse commuting graph for dihedral groups of order 18 up to 32, i.e. for both even and odd *n*.

Definition 3.1 Inverse Commuting Elements
These are the sets of non-central elements of a group that commute and are inverses of each other. i.e. for distinct $x, y \in D_{2n}$. $xy = yx = 1$, where 1 is the identity element.

Proposition 3.2
The set of inverse commuting elements of a dihedral group is given by $\{a^i,\ a^{n-i}\}$, $\ i = 1, 2, 3, \ldots, \alpha$ where

$$\alpha = \begin{cases} \frac{n-2}{2}, & for\ \ n\ even \\ \frac{n-1}{2}, & for\ \ n\ \ odd \end{cases}$$

Proof
In a dihedral group (D_{2n}), two distinct reflections commute only for even *n*, and no two of such reflections that commute are inverses of each other. However, for rotations, they commute for both even and odd *n*. It is clear that two distinct rotations which commute are inverses of each other if they result to the identity,

i.e. $\forall\, a^i, a^{n-i} \in D_{2n}$

$$a^i * a^{n-i} = a^{i+n-i} = a^n = 1.$$

Note that α gives the number of sets of the inverse commuting elements.

Definition 3.3 Inverse Commuting Graph
A graph G^{IC} is called inverse commuting graph if the vertex set is the non-central elements that commute and are inverses of each other, two distinct vertices x and *y* are adjacent if $xy = yx = 1$, where 1 is the identity element.

Theorem 3.4
Let $D_{18} = < a, b : a^9 = b^2 = 1,\ bab^{-1} = a^{-1} >$,
then the inverse commuting graph for D_{18} is

$$G_{18}^{IC} = \cup_{i=1}^{4} K_2.$$

Proof
Since $D_{18} = D_{2(9)}, n = 9$.
Applying Proposition 3.2, we get $\alpha = 4$ and the sets of inverse commuting elements as

$\{a, a^8\}, \{a^2, a^7\}, \{a^3, a^6\}, \{a^4, a^5\}$.

Then, the inverse commuting graph for D_{18} is a union of four identical complete graphs K_2. i.e. $G_{18}^{IC} = \cup_{i=1}^{4} K_2$.

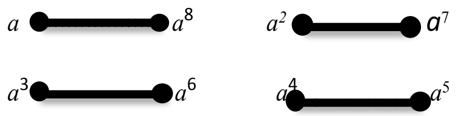

Fig. 1 Inverse commuting graph of D_{18}.

Theorem 3.5
Let $D_{20} = < a, b : a^{10} = b^2 = 1, bab^{-1} = a^{-1} >$,
then the inverse commuting graph for D_{20} is $G_{20}^{IC} = \cup_{i=1}^{4} K_2$

Proof
Since $D_{20} = D_{2(10)}$, so $n = 10$.
Applying Proposition 3.2, we get $\alpha = 4$ and sets of inverse commuting elements as

$\{a, a^9\}, \{a^2. a^8\}, \{a^3, a^7\}, \{a^4, a^6\}$.

Then, the inverse commuting graph for D_{20} is a union of four identical complete graphs K_2. i.e. $G_{20}^{IC} = \cup_{i=1}^{4} K_2$.

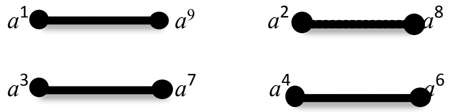

Fig. 2 Inverse commuting graph of D_{20}.

Theorem 3.6
Let $D_{22} = < a, b :\ a^{11} = b^2 = 1,\ bab^{-1} = a^{-1} >$,
then the inverse commuting graph for D_{22} is $G_{22}^{IC} = \cup_{i=1}^{5} K_2$.

Proof
Since $D_{22} = D_{2(11)}$, so $n = 11$.
Applying Proposition 3.2, we get$\alpha = 5$ and sets of inverse commuting elements as $\{a, a^{10}\}, \{a^2, a^9\}, \{a^3, a^8\}, \{a^4, a^7\}, \{a^5, a^6\}$. Then, the inverse commuting graph for D_{22} is a union of five identical complete graphs K_2 i.e. $G_{22}^{IC} = \cup_{i=1}^{5} K_2$.

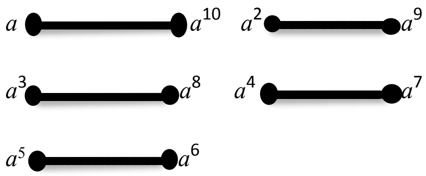

Fig. 3 Inverse commuting graph of D_{22}.

Theorem 3.7
Let $D_{24} = < a, b : a^{12} = b^2 = 1, bab^{-1} = a^{-1} >$,
then the inverse commuting graph for D_{24} is $G_{24}^{IC} = \cup_{i=1}^{5} K_2$.

Proof
Since $D_{24} = D_{2(12)}$, so $n = 12$.
Applying Proposition 3.2, we get $\alpha = 5$ and sets of inverse commuting elements as

$\{a, a^{11}\}, \{a^2, a^{10}\}, \{a^3, a^9\}, \{a^4, a^8\}, \{a^5, a^7\}$.

Then, the inverse commuting graph for D_{24} is a union of five identical complete graphs k_2 i.e. $G_{24}^{IC} = \cup_{i=1}^{5} K_2$.

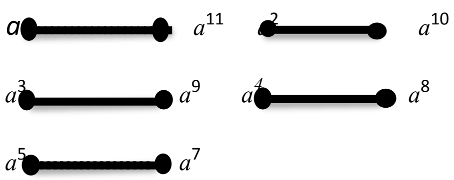

Fig. 4 Inverse commuting graph of D_{24}.

Theorem 3.8
Let $D_{26} =< a, b : a^{13} = b^2 = 1, bab^{-1} = a^{-1} >$,
then the inverse commuting graph for D_{26} is $G_{26}^{IC} = \cup_{i=1}^{6} k_2$.

Proof
Since $D_{26} = D_{2(13)}$, so $n = 13$.

Applying Proposition 3.2, we get $\alpha = 6$ and sets of inverse commuting elements as

$$\{a, a^{12}\}, \{a^2, a^{11}\}, \{a^3, a^{10}\}, \{a^4, a^9\}, \{a^5, a^8\}, \{a^6, a^7\}.$$

Then, the inverse commuting graph for D_{26} is a union of six identical complete graphs K_2 i.e. $G_{26}^{IC} = \cup_{i=1}^{6} K_2$.

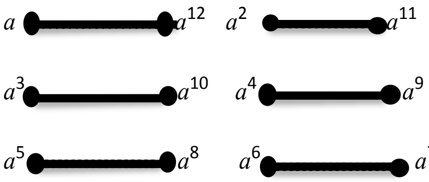

Fig. 5 Inverse commuting graph of D_{26}.

Theorem 3.9
Let $D_{28} =< a, b: a^{14} = b^2 = 1, bab^{-1} = a^{-1} >$,
then the inverse commuting graph for D_{28} is $G_{28}^{IC} = \cup_{i=1}^{6} K_2$.

Proof
Since $D_{28} = D_{2(14)}$, so $n = 14$.
Applying Proposition 3.2, we get $\alpha = 6$ and sets of inverse commuting elements as

$$\{a, a^{13}\}, \{a^2, a^{12}\}, \{a^3, a^{11}\}, \{a^4, a^{10}\}, \{a^5, a^9\}, \{a^6, a^8\}.$$

Then, the inverse commuting graph for D_{28} is $G_{28}^{IC} = \cup_{i=1}^{6} K_2$

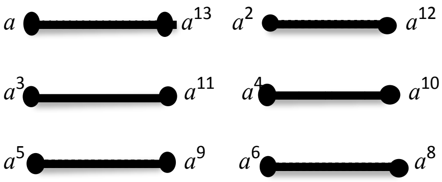

Fig. 6 Inverse commuting graph of D_{28}.

Theorem 3.10
Let $D_{30} =< a, b, a^{15} = b^2 = 1, bab^{-1} = a^{-1} >$,
then the inverse commuting graph for D_{30} is $G_{30}^{IC} = \cup_{i=1}^{7} K_2$

Proof
Since $D_{30} = D_{2(15)}$, so $n = 15$.
Applying Proposition 3.2, we get $\alpha = 7$ and sets of inverse commuting elements as

$$\{a, a^{14}\}, \{a^2, a^{13}\}, \{a^3, a^{12},\}, \{a^4, a^{11}\}, \{a^5, a^{10}\}, \{a^6, a^9\}, \{a^7, a^8\}.$$

Then, the inverse commuting graph for D_{30} is $G_{30}^{IC} = \cup_{i=1}^{7} K_2$

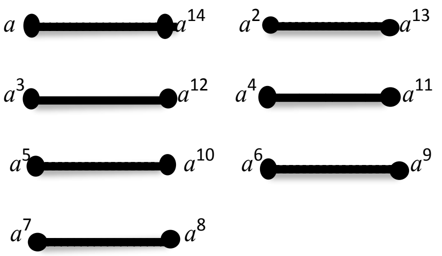

Fig. 7 Inverse commuting graph of D_{30}.

Theorem 3.11
Let $D_{32} =< a, b: a^{16} = b^2 = 1, bab^{-1} = a^{-1} >$,
then the inverse commuting graph for D_{32} is $G_{32}^{IC} = \cup_{1}^{7} K_2$.

Proof
Since $D_{32} = D_{2(16)}$, so $n = 16$.
Applying Proposition 3.2 gives $\alpha = 7$ and sets of inverse commuting elements as

$$\{a, a^{15}\}, \{a^2, a^{14}\}, \{a^3, a^{13}\}, \{a^4, a^{12}\}, \{a^5, a^{11}\}, \{a^6, a^{10}\}, \{a^7, a^9\}.$$

Then, the inverse commuting graph for D_{32} is $G_{32}^{IC} = \cup_{i=1}^{7} K_2$

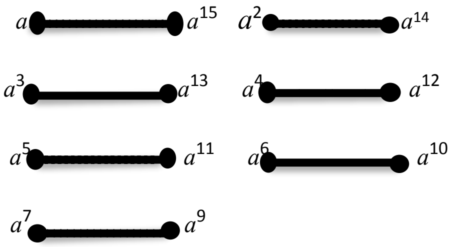

Fig. 8 Inverse commuting graph of D_{32}.

The independence polynomial of inverse commuting graphs of dihedral groups of orders up to 20

In this part, we study the independence polynomials of the inverse commuting graphs for dihedral groups of orders up to 20, using Hoede and Li's method [6] where we got the independent sets and independence numbers of the graphs. Here, we show the computation for D_{14} and D_{20} representing even and odd values of *n*.

Theorem 4.1
Let $D_{14} =< a, b : a^7 = b^2 = 1, bab^{-1} = a^{-1} >$,
the independence polynomial of the inverse commuting graph for D_{14} is

$$I(G_1^{IC}, x) = 1 + 6x + 12x^2 + 8x^3.$$

Proof
By Definitions 2.1 and 2.2, the inverse commuting graph G_{14}^{IC} has 8 independent sets of order 3, 12 of order 2, and 6 of order 1 as follows:

Order 3

$$\{a, a^2, a^3\}, \{a, a^2, a^4\}, \{a, a^5, a^3\}, \{a, a^5, a^4\},$$

$$\{a^2, a^6, a^3\}, \{a^6, a^2, a^4\}, \{a^6, a^5, a^3\}, \{a^6, a^5, a^4\}.$$

Order 2

$$\{a, a^2\}, \{a, a^5\}, \{a, a^3\}, \{a, a^4\}, \{a^6, a^2\}, \{a^6, a^5\},$$

$$\{a^6, a^3\}, \{a^6, a^4\}, \{a^2, a^3\}, \{a^2, a^4\}, \{a^5, a^3\}. \{a^5, a^4\}.$$

Order 1

$$\{a\}, \{a^2\}, \{a^3\}, \{a^4\}, \{a^5\}, \{a^6\}.$$

The independence number, $n(G_{18}^{IC}) = 3$.

By Definition 2.3

$$\mathrm{I}(G_{14;}^{\mathrm{IC}} x) = \sum_{k=0}^{3} C_k x^k$$
$$= C_0 x^0 + C_1 x^1 + C_2 x^2 + C_3 x^3$$
$$= 1 + 6x + 12x^2 + 8x^3.$$

Theorem 4.2
Let $D_{20} =< a, b : a^{10} = b^2 = 1, bab^{-1} = a^{-1} >$,
the independence polynomial of inverse commuting graph D_{20} is

$$I(G^{IC}_{20;}x) = 1 + 8x + 24x^2 + 32x^3 + 16x^4.$$

Proof
By Definitions 2.1 and 2.2, the inverse commuting graph G^{IC}_{20} has 16 independent sets of order 4, 32 of order 3, 24 of order 2, and 9 of order 1 as follows:

$$\{a, a^2, a^3, a^4\}, \{a, a^2, a^3, a^6\}, \{a, a^2, a^7, a^4\}, \{a, a^2, a^7, a^6\},$$

$$\{a, a^8, a^3, a^4\}, \{a, a^8, a^3, a^6\}, \{a, a^8, a^7, a^4\}, \{a, a^8, a^7, a^6\},$$

$$\{a^9, a^2, a^3, a^4\}, \{a^9, a^2, a^3, a^6\}, \{a^9, a^2, a^7, a^4\}, \{a^9, a^2, a^7, a^6\},$$

$$\{a^9, a^8, a^3, a^4\}, \{a^9, a^8, a^3, a^6\}, \{a^9, a^8, a^7, a^4\}, \{a^9, a^8, a^7, a^6\};$$

Order 3

$$\{a,, a^2, a^3\}, \{a, a^2, a^7\}, \{a, a^8, a^3\}, \{a, a^8, a^7\}, \{a, a^2, a^4\}, \{a, a^2, a^6\},$$

$$\{a, a^8, a^4\}, \{a, a^8, a^6\}, \{a^9, a^2, a^3\}, \{a^9, a^2, a^7\}, \{a^9, a^8, a^3\}, \{a^9, a^8, a^7\},$$

$$\{a^9, a^2, a^4\}, \{a^9, a^2, a^6\}, \{a^9, a^8, a^4\}, \{a^9, a^8, a^6\}, \{a, a^3, a^4\}, \{a, a^3, a^6\},$$

$$\{a, a^7, a^4\}, \{a, a^7, a^6\}, \{a^9, a^3, a^4\}, \{a^9, a^3, a^6\}, \{a^9, a^7, a^4\}, \{a^9, a^7, a^6\},$$

$$\{a^2, a^3, a^4\}, \{a^2, a^3, a^6\}, \{a^2, a^7, a^4\}, \{a^2, a^7, a^6\}, \{a^8, a^3, a^4\}, \{a^8, a^3, a^6\},$$

$$\{a^8, a^7, a^4\}, \{a^8, a^7, a^6\};$$

Order 2

$$\{a, a^2\}, \{a, a^8\}, \{a, a^3\}, \{a, a^7\}, \{a, a^4\}, \{a, a^6\},$$

$$\{a^9, a^2\}, \{a^9, a^8\}, \{a^9, a^3\}, \{a^9, a^7\}, \{a^9, a^4\}, \{a^9, a^6\},$$

$$\{a^2, a^3\}, \{a^2, a^7\}, \{a^2, a^4\}, \{a^2, a^6\}, \{a^8, a^3\}, \{a^8, a^7\},$$

$$\{a^8, a^4\}, \{a^8, a^6\}, \{a^3, a^4\}, \{a^3, a^6\}, \{a^7, a^4\}, \{a^7, a^6\};$$

Order 1

$$\{a\}, \{a^2\}, \{a^3\}, \{a^4\}, \{a^5\}, \{a^6\}, \{a^7\}, \{a^8\}, \{a^9\}.$$

The independence number, $n(G^{IC}_{20}) = 4$.

By Definition 2.3

$$I(G^{IC}_{20}) = \sum_{k=0}^{4} C_k x^k$$
$$= C_0x^0 + C_1x^1 + C_2x^2 + C_3x^3 + C_4x^4$$
$$= 1 + 8x + 24x^2 + 32x^3 + 16x^4.$$

Table 1 shows the independence polynomials of inverse commuting graph for dihedral groups of order 6 to 20 found using Hoede and Li's concept.

Table 1 Independence Polynomials using Hoede and Li's concept.

S/N	D_{2n}	Independence Polynomial
1	D_6	$1 + 2x$
2	D_8	$1 + 2x$
3	D_{10}	$1 + 4x + 4x^2$
4	D_{12}	$1 + 4x + 4x^2$
5	D_{14}	$1 + 6x + 12x^2 + 8x^3$
6	D_{16}	$1 + 6x + 12x^2 + 8x^3$
7	D_{18}	$1 + 8x + 24x^2 + 32x^3 + 16x^4$
8	D_{20}	$1 + 8x + 24x^2 + 32x^3 + 16x^4$

Independence polynomial using formula

As the order of dihedral group increases, it becomes cumbersome to generate the independent sets. This section shows how to compute independence polynomial without actually getting the independent sets. This is done via a pattern recognition for the number of independent sets which serve as coefficients in the polynomials. An observation of the pattern of the computed independence polynomials of dihedral groups of orders up to 20 gives the following Lemma.

Lemma 5.1
Let D_{2n} be a dihedral group and G^{IC}_{2n} be the inverse commuting graph for the group, the independence polynomial is given by

$$I(G^{IC}_{2n}; x) = \sum_{i=0}^{\alpha} (2^{\alpha-1})\ {}^{\alpha}C_{(\alpha-i)} x^{\alpha-i}$$

where

$$\alpha = \begin{cases} \dfrac{n-2}{2}, & n \text{ even} \\ \dfrac{n-1}{2}, & n \text{ odd} \end{cases}$$

Proof
Note that the number of independent sets of order $(\alpha - i)$ in the graph is $2^{(\alpha-i)}\ {}^{\alpha}C_{(\alpha-i)}$ where 2 represents a pair of non-adjacent vertices in the graph with α as the maximum cardinality in the independent sets. This means $0 \leq i \leq \alpha$.

By combinatorial approach, $\alpha - i$ vertices are selected from α, with α also representing the number of complete graphs K_2. For $i = 0$ the number of independent set is 2^{α}, for $i = \alpha$ the number of independent set is 1, and for $0 < i < \alpha$ the number of independent sets is $2^{(\alpha-i)}\ {}^{\alpha}C_{(\alpha-i)}$.So, the product of $2^{(\alpha-i)}$ and ${}^{\alpha}C_{(\alpha-i)}$ gives the number of independent sets of order $(\alpha - i)$ which serve as coefficients to the desired polynomial.

Using the Lemma given, obtaining the independence polynomial of inverse commuting graph for D_6 to D_{20} gives the same result as in Section 4 using Hoede and Li's method. For instance, the computation for D_{18} and D_{20} is shown in Theorems 5.2 and 5.3

Theorem 5.2
Let $D_{18} =< a, b : a^9 = b^2 = 1, bab^{-1} = a^{-1} >$,
then the independence polynomial of the inverse commuting graph for D_{18} is

$$I(G^{IC}_{18}; x) = 1 + 8x + 24x^2 + 32x^3 + 16x^4.$$

Proof
By Lemma 5.1, $n = 9$, $\alpha = 4$

$$I(G^{IC}_{18}; x) = \sum_{i=0}^{4} (2^{4-i})\ ({}^{4}C_{4-i}) x^{4-i}$$

$$=2^4({}^4C_4)x^4+2^3({}^4C_3)x^3+2^2({}^4C_2)x^2+2^1({}^4C_1)x^2+2^0({}^4C_0)x^0$$

$$= 16x^4+32x^3+24x^2+8x+1.$$

Theorem 5.3
Let $D_{20} =< a, b : a^{10} = b^2 = 1, bab^{-1} = a^{-1} >$,
then the independence polynomial of the inverse commuting graph for D_{20}

$$I(G^{IC}_{20}; x) = 1 + 8x + 24x^2 + 32x^3 + 16x^4.$$

Proof
By Lemma 5.1, $n = 10$, $\alpha = 4$,

$$I(G^{IC}_{20}; x) = \sum_{i=0}^{4} (2^{4-i})\ ({}^{4}C_{4-i}) x^{4-i}$$

$$=2^4({}^4C_4)x^4+2^3({}^4C_3)x^3+2^2({}^4C_2)x^2+2^1({}^4C_1)x^2+2^0({}^4C_0)x^0$$

$= 16x^4+32x^3+24x^2+8x+1$.

For the next six consecutive orders of the dihedral groups, we have the following Theorems.

Theorem 5.4
Let $D_{22} =< a, b : a^{11} = b^2 = 1, bab^{-1} = a^{-1} >$,
then the independence polynomial of the inverse commuting graph for D_{22} is

$$I(G_{22}^{IC}; x) = 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5.$$

Proof
By Lemma 5.1, $n = 11$, $\alpha = 5$,

$I(G_{22}^{IC}; x) = \sum_{i=0}^{5}(2^{5-i})\,({}^5C_{5-i})x^{5-i}$

$=2^5({}^5C_5)x^5+2^4({}^5C_4)x^4+2^3({}^5C_3)x^3+2^2({}^5C_2)x^2+2^1({}^5C_1)x^1+2^0({}^5C_0)x^0$

$= 32x^5+80x^4+80x^3+40x^2+10x+1$.

Theorem 5.5
Let $D_{24} =< a, b : a^{12} = b^2 = 1, bab^{-1} = a^{-1} >$,
then the independence polynomial of the inverse commuting graph for D_{24} is

$$I(G_{24}^{IC}; x) = 1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5.$$

Proof
By Lemma 5.1 $n = 12$, $\alpha = 5$,

$I(G_{24}^{IC}; x) = \sum_{i=0}^{5}(2^{5-i})\,({}^5C_{5-i})x^{5-i}$

$=2^5({}^5C_5)x^5+2^4({}^5C_4)x^4+2^3({}^5C_3)x^3+2^2({}^5C_2)x^2+2^1({}^5C_1)x^1+2^0({}^5C_0)x^0$

$= 32x^5+80x^4+80x^3+40x^2+10x+1$.

Theorem 5.6
Let $D_{26} =< a, b : a^{13} = b^2 = 1, bab^{-1} = a^{-1} >$,
then the independence polynomial of the inverse commuting graph for D_{26} is

$$I(G_{26}^{IC}; x) = 1 + 12x + 60x^2 + 160x^3 + 240x^4 + 190x^5 + 64x^6.$$

Proof
By Lemma 5.1, $n = 13$, $\alpha = 6$,

$I(G_{26}^{IC}; x) = \sum_{i=0}^{6}(2^{6-i})\,({}^6C_{6-i})x^{6-i}$

$=2^6({}^6C_6)x^6+2^5({}^6C_5)x^5+2^4({}^5C_4)x^4+2^3({}^6C_3)x^3+2^2({}^6C_2)x^2+2^1({}^6C_1)x^1+$

$2^0({}^6C_0)x^0$

$= 64x^6+190x^5+240x^4+160x^3+60x^2+12x+1$.

Theorem 5.7
Let $D_{28} =< a, b : a^{14} = b^2 = 1, bab^{-1} = a^{-1} >$,
then the independence polynomial of the inverse commuting graph for D_{28} is

$$I(G_{28}^{IC}; x) = 1 + 12x + 60x^2 + 160x^3 + 240x^4 + 190x^5 + 64x^6.$$

Proof
By Lemma 5.1, $n = 14$, $\alpha = 6$,

$I(G_{28}^{IC}; x) = \sum_{i=0}^{6}(2^{6-i})\,({}^6C_{6-i})x^{6-i}$

$=2^6({}^6C_6)x^6+2^5({}^6C_5)x^5+2^4({}^5C_4)x^4+2^3({}^6C_3)x^3+2^2({}^6C_2)x^2+2^1({}^6C_1)x^1+$

$2^0({}^6C_0)x^0$

$= 64x^6+190x^5+240x^4+160x^3+60x^2+12x+1$.

Theorem 5.8
Let $D_{30} =< a, b : a^{15} = b^2 = 1, bab^{-1} = a^{-1} >$,
then the independence polynomial of the inverse commuting graph for D_{30} is

$$I(G_{30}^{IC}; x) = 1 + 14x + 84x^2 + 280x^3 + 560x^4 + 672x^5 + 448x^6 + 128x^7.$$

.

Proof
By Lemma 5.1, $n = 15$, $\alpha = 7$,

$I(G_{30}^{IC}; x) = \sum_{i=0}^{7}(2^{7-i})\,({}^7C_{7-i})x^{7-i}$

$=2^7({}^7C_7)x^7+2^6({}^7C_6)x^6+2^5({}^7C_5)x^5+2^4({}^7C_4)x^4+2^3({}^7C_3)x^3+2^2({}^7C_2)x^2$

$+2^1({}^7C_1)x^1+2^0({}^7C_0)x^0$

$=448x^6+672x^5+560x^4+280x^3+84x^2+14x+1$.

Theorem 5.9
Let $D_{32} =< a, b : a^{16} = b^2 = 1, bab^{-1} = a^{-1} >$,
then the independence polynomial of the inverse commuting graph for D_{32} is

$$I(G_{32}^{IC}; x) = 1 + 14x + 84x^2 + 280x^3 + 560x^4 + 672x^5 + 448x^6 + 128x^7$$

Proof
By Lemma 5.1, $n = 16$, $\alpha = 7$,

$I(G_{32}^{IC}; x) = \sum_{i=0}^{7}(2^{7-i})\,({}^7C_{7-i})x^{7-i}$

$=2^7({}^7C_7)x^7+2^6({}^7C_6)x^6+2^5({}^7C_5)x^5+2^4({}^7C_4)x^4+$

$2^3({}^7C_3)x^3+2^2({}^7C_2)x^2+2^1({}^7C_1)x^1+2^0({}^7C_0)x^0$

$=448x^6+672x^5+560x^4+280x^3+84x^2+14x+1$.

CONCLUSION

The inverse commuting graphs for dihedral groups of order 18 up to 32 were found. It was observed that the inverse commuting graph for a dihedral group D_{2n} is given by

$$G_{2n}^{IC} = \bigcup_{i=1}^{\alpha}(K_2)$$

where

$$\alpha = \begin{cases} \frac{n-2}{2}, & n \text{ even} \\ \frac{n-1}{2}, & n \text{ odd} \end{cases}$$

The independence polynomials for the graphs were found and a formula for getting the independence polynomials for the inverse commuting graphs of dihedral groups of order up to 32 was also found to be

$$I(G_{2n}^{IC}; x) = \sum_{i=0}^{\alpha}(2^{\alpha-i})\binom{\alpha}{\alpha-i}x^{\alpha-i},$$

where

$$\alpha = \begin{cases} \frac{n-2}{2}, & n \text{ even} \\ \frac{n-1}{2}, & n \text{ odd} \end{cases}$$

The independence polynomials occur in identical pairs, i.e. groups with the same α having the same inverse commuting graphs and consequently, having the same independence polynomial as shown in Table 2.

Table 2 Independence polynomials using Lemma 5.1.

S/N	D_{2n}	Independence Polynomial
1	D_{18}	$1+8x+24x^2+32x^3+16x^4$
2	D_{20}	$1+8x+24x^2+32x^3+16x^4$
3	D_{22}	$1+10x+40x^2+80x^3+80x^4+32x^5$
4	D_{24}	$1+10x+40x^2+80x^3+80x^4+32x^5$
5	D_{26}	$1+12x+60x^2+160x^3+240x^4+190x^5+64x^6$
6	D_{28}	$1+12x+60x^2+160x^3+240x^4+190x^5+64x^6$
7	D_{30}	$1+14x+84x^2+280x^3+560x^4+672x^5+448x^6+128x^7$
8	D_{32}	$1+14x+84x^2+280x^3+560x^4+672x^5+448x^6+128x^7$

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