The independence and clique polynomial of the conjugacy class graph of dihedral group

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INTRODUCTION

In order to understand this research, few basic concepts from graph theory are first stated here. A graph is $G=(V,E)$ containing $V$ as a nonempty set of vertices and $E$ as a set of unordered pair of elements of $V$, called the edges. Two vertices $u, v \in V$ are adjacent to each other in $G$ if and only if there is an edge between $u$ and $v$, i.e. $(u, v) \in E$. A vertex $v$ is an isolated vertex if it is not adjacent to any other vertices $u$ in the graph. An edge $e=(x,y)$ is called incident with each one of its end vertices, $x$ and $y$. Note that, only simple graphs is considered throughout this study. A graph is simple if it has no loops and no multiple edges.

A vertex $u$ is a neighbor of a vertex $v$ in $G$ if $(u,v)$ is an edge of $G$. Open neighborhood (or just neighborhood), of $v$ is defined to be the set of all vertices adjacent to $v$, denoted as $N(v)=\{u \in V | (u,v) \in E, u \neq v\}$. The set $N(v)=N(v) \cup \{v\}$ is the closed neighborhood of $v$ in $G$ [1]. If the neighborhood of every vertex is empty, means that there is no edge in the graph, then the graph is called empty graph on $n$ vertices, denoted by $E_n$. If $n=0$, then the graph is called null graph, denoted by $E_0=\emptyset$, and if we have $n=1$, the graph $E_1$ is called singleton, a single vertex graph.

The complement of an empty graph is the complete graph, $K_n$. It is a graph with $n$ vertices where each pair of distinct vertices is connected by an edge [2].

This paper is structured as follows: the first part is the introduction, the second part is the methodology used in this research while the third part includes the main results. The independence polynomial and clique polynomial are computed for the conjugacy class graph of the dihedral group of order $2n$, with group representation

$$D_{2n} = \{a,b : a^n = b^2 = 1, bab = a^{-1}\}$$

where $n \geq 3$, $n \in \mathbb{N}$.
PRELIMINARIES

Some basic concepts in graph theory that are related to group theory are included in this section. The independence and clique polynomial are determined by using some properties that will be stated here.

The following are some basic concepts on independence polynomial, that are used throughout this paper.

Definition 2.1 [3] Independent Set, Independence Number
An independent set of a graph is a set of vertices such that no two distinct vertices are adjacent. The independence number of the graph, denoted by \( \alpha(G) \), is the maximum number of vertices in an independent set of the graph.

Definition 2.2 [3] Independence Polynomial
The independence polynomial of a graph \( \Gamma \) is the polynomial whose coefficient on \( x^k \) is given by the number of independent sets of order \( k \) in \( \Gamma \). This is denoted by \( I(\Gamma;x) \). So

\[
I(\Gamma;x) = \sum_{k=0}^{\infty} \alpha_k x^k
\]

where \( \alpha_k \) is the number of independent sets of size \( k \) and \( \infty \) is the independence number of \( \Gamma \).

Theorem 2.1 [3]
Let \( \Gamma_1 \) and \( \Gamma_2 \) be two disjoint graphs. Then

\[
I(\Gamma_1 \cup \Gamma_2; x) = I(\Gamma_1; x) \cdot I(\Gamma_2; x).
\]

From the definition and theorem above, Ferrin [4] had applied them in obtaining the independence polynomial of some common graphs. Below are the propositions that will be used in proving our results.

Proposition 2.1 [4]
The independence polynomial of an empty graph on \( n \) vertices, is

\[
I(E_n; x) = I(K_1; x) = 1 + nx.
\]

Proposition 2.2 [4]
The independence polynomial of a complete graph on \( n \) vertices is

\[
I(K_n; x) = 1 + nx.
\]

Next, in the following are the basic concepts on clique polynomial, that will be used throughout this paper.

Definition 2.3 [3] Clique Polynomial
A clique of a graph is a set of vertices in which every vertex is adjacent to every other vertex. The clique number of the graph, denoted by \( \omega(G) \), is the size of the biggest clique.

Definition 2.4 [3] Clique Polynomial
The clique polynomial of a graph \( \Gamma \) is the polynomial whose coefficient on \( x^k \) is given by the number of cliques of order \( k \) in \( \Gamma \). This is denoted by \( C(\Gamma;x) \). So

\[
C(\Gamma;x) = \sum_{k=0}^{\infty} \omega_k x^k
\]

where \( \omega_k \) is the number of cliques of order \( k \) and \( \infty \) is the clique number of \( \Gamma \).

Theorem 2.2 [3]
Let \( \Gamma_1 \) and \( \Gamma_2 \) be two disjoint graphs. Then

\[
C(\Gamma_1 \cup \Gamma_2; x) = C(\Gamma_1; x) \cdot C(\Gamma_2; x).
\]

Haedo and Li [3] had also applied Definition 2.4 and Theorem 2.2 to obtain the following propositions.

Proposition 2.3 [3]
The clique polynomial of an empty graph on \( n \) vertices, is

\[
C(E_n; x) = C(K_1; x) = 1 + nx.
\]

Proposition 2.4 [3]
The clique polynomial of a complete graph on \( n \) vertices is

\[
C(K_n; x) = (1 + x)^n.
\]

Lastly, we state some basic definitions and results from group theory and graph theory that are related to conjugacy class graph.

Definition 2.5 [3] Conjugacy Class
Let \( G \) be a group and \( a, x \in G \). For a fixed element \( a \in G \), the conjugacy class of \( a \) in \( G \) is given as

\[
cl(a) = \{ g \in G : \text{there exists } x \in G, g = xax^{-1} \}.
\]

Also, \( xax^{-1} \) is called the conjugate of \( a \) by \( x \) in the group \( G \).

Proposition 2.5 [5]
Let \( a \) be an element in \( G \). If the conjugacy class of \( a \) contains only one element, then \( a \) lies in the center of the group, \( Z(G) \).

Theorem 2.3 [6]
Let \( D_{2n} \) be dihedral groups of order \( 2n \), then the conjugacy classes in \( D_{2n} \), depending on the parity of \( n \), are as follows.

1. For odd \( n \):

\[
\{1\}, \{a, a^{-1}\}, \{a, a^{-2}\}, ..., \{a^{n-2}, a^{n-1}\} \text{ and } \{a^i : 0 \leq i \leq n-1\}
\]

2. For even \( n \):

\[
\{1\}, \{a, a^{-1}\}, \{a, a^{-2}\}, ..., \{a^{n-2}, a^{n-1}\}, \{a^n\}.
\]

\[
\{a^{2i} : 0 \leq i \leq \frac{n}{2} - 1\} \text{ and } \{a^{2i+1} : 0 \leq i \leq \frac{n}{2} - 1\}
\]

Definition 2.6 [7] Conjugacy Class Graph
A conjugacy class graph \( G' \) of a group \( G \), is defined as the graph whose vertex set, \( V(G') \), is the non-central conjugacy classes of \( G \), in which two distinct vertices \( cl(a) \) and \( cl(b) \) are adjacent if and only if their class cardinalities are not coprime i.e. \( |cl(a)| \cdot |cl(b)| > 1 \).

Theorem 2.4 [8]
Let \( D_{2n} \) be dihedral groups of order \( 2n \), then the conjugacy class graphs of \( D_{2n} \) can be stated as in the following.

Case 1: \( n \) is odd

\[
\Gamma_{D_{2n}} = K_{\frac{n+1}{2}} \text{ such that } K_{\frac{n+1}{2}} \text{ contains vertices } cl(a), cl(a^2), ..., \text{ and } cl\left(a^{\frac{n+1}{2}}\right), \text{ and } K_{\frac{n+1}{2}} \text{ is the isolated vertex } cl(b).
\]

Case 2: \( n \) and \( \frac{n}{2} \) are even
\[ \frac{\alpha}{D_n} = K_{\frac{n+2}{2}}, \] such that it contains vertices
\[ cl(a), cl(a^2), \ldots, cl\left( a^\frac{n}{2} \right), cl(b), \] and \( cl(ab) \).

Case 3: \( n \) is even and \( \frac{n}{2} \) is odd
\[ \frac{\alpha}{D_n} = K_{\frac{n}{2}} \] such that \( K_{\frac{n}{2}} \) contains vertices \( cl(a) \),
\[ cl(a^2), \ldots, cl\left( a^\frac{n}{2} \right), \] and \( K_1 \) contains vertices \( cl(b) \) and \( cl(ab) \).

The aim of this paper is to obtain the independence and clique polynomial of the conjugacy class graph for the dihedral groups \( D_{2n} \) of order \( 2n \).

**MAIN RESULTS**

This section consists of two parts. The first part presents the result on the independence polynomial of the conjugacy class graph of \( D_{2n} \) while the second part presents the clique polynomial of the conjugacy class graph of \( D_{2n} \).

The independence polynomial of conjugacy class graph of dihedral groups

This is the first part of the main result in which the independence polynomials of the conjugacy class graph of \( D_{2n} \) are obtained, depending on the parity of \( n \).

**Theorem 3.1** Suppose that \( D_{2n} \) be dihedral groups of order \( 2n \) where \( n \geq 3, n \in \mathbb{N} \) then the independence polynomials of the conjugacy class graphs of \( D_{2n} \) are as follows.

\[
I\left( \frac{\alpha}{D_n} : x \right) = \begin{cases} 
1+ (\frac{n+1}{2})x + (\frac{n-1}{2})x^2 & ; \text{n is odd} \\
1+ (\frac{n+2}{2})x + (\frac{n+1}{2})(n-1)x^2 & ; \text{n and } \frac{n}{2} \text{ are even} \\
1+(\frac{n}{2})x+(n-2)x^2 & ; \text{n is even and } \frac{n}{2} \text{ is odd}. 
\end{cases}
\]

**Proof** Let \( D_{2n} \) be dihedral groups of order \( 2n \) and \( \frac{\alpha}{D_n} \) be its conjugacy class graph.

Case 1: \( n \) is odd
From Theorem 2.4, the conjugacy class graph of \( D_{2n} \) is \( K_{\frac{n+2}{2}} \).

By Theorem 2.1 and Proposition 2.2, we can compute the independence polynomial of \( \frac{\alpha}{D_n} \) as follows:

\[
I\left( \frac{\alpha}{D_n} : x \right) = I(K_{\frac{n+2}{2}} \cup K_1; x) \\
= I(K_{\frac{n+2}{2}}; x) - I(K_1; x) \]

\[
= 1 + (\frac{n}{2})x + (\frac{n}{2})x^2. 
\]

Case 2: \( n \) and \( \frac{n}{2} \) are even
From Theorem 2.4, the conjugacy class graph of \( D_{2n} \) is \( K_{\frac{n+2}{2}} \).

By Proposition 2.2, the independence polynomial of \( \frac{\alpha}{D_n} \) can be computed as follows:

\[
I\left( \frac{\alpha}{D_n} : x \right) = I(K_{\frac{n+2}{2}}; x) \\
= 1 + (\frac{n+2}{2})x + (\frac{n+1}{2})x^2. 
\]

Case 3: \( n \) is even and \( \frac{n}{2} \) is odd
From Theorem 2.4, the conjugacy class graph of \( D_{2n} \) is \( K_{\frac{n}{2}} \).

By Theorem 2.1 and Proposition 2.2, the independence polynomial of \( \frac{\alpha}{D_n} \) can be computed as follows:

\[
I\left( \frac{\alpha}{D_n} : x \right) = I(K_{\frac{n+2}{2}} \cup K_1; x) \\
= I(K_{\frac{n+2}{2}}; x) - I(K_1; x) \]

\[
= 1 + (\frac{n-2}{2})x + (\frac{n-1}{2})x^2. 
\]

**Example 3.1** Let \( G \) be the dihedral group of order 14 \((n = 7)\), i.e. \( G = D_{14} = \{ a, b : a^7 = b^2 = 1, bab = a^{-1} \} \).

By Theorem 2.3, the conjugacy classes of \( D_{14} \) are \( cl(e) = \{ a, a^7 \} \), \( cl(a) = \{ a^2, a^6 \} \), \( cl(a^2) = \{ a^3, a^5 \} \), and \( cl(b) = \{ b, ab, a' b \} \).

By Theorem 2.4, if \( \frac{\alpha}{D_n} \) is the conjugacy class graph of \( D_{14} \) with the set of vertices, \( V\left( \frac{\alpha}{D_n} \right) = \{ cl(a), cl(a^2), cl(a^3), cl(b) \} \), then \( \frac{\alpha}{D_n} = K_{\frac{7}{2}} \cup K_1 \).

Hence, the independence polynomial of the conjugacy class graph of \( D_{14} \) is

\[
I\left( \frac{\alpha}{D_n} : x \right) = I(K_{\frac{7}{2}} \cup K_1; x) \\
= 1 + (\frac{7+1}{2})x + (\frac{7+1}{2})x^2 \]

\[
= 1 + 2x + 3x^2. 
\]
The clique polynomial of conjugacy class graph of dihedral groups

The second part of our main result is the clique polynomial of the conjugacy class graph of \( D_{2n} \).

**Theorem 3.2** Suppose that \( D_{2n} \) be dihedral groups of order \( 2n \) where \( n \geq 3, n \in \mathbb{N} \) then the clique polynomials of the conjugacy class graphs of \( D_{2n} \) are as follows.

\[
C\left( \Gamma_{\mathcal{D}_{2n}}^\mathcal{G} ; x \right) = \begin{cases} 
  x + (1 + x)^{n + 1} & : n \text{ is odd} \\
  (1 + x)^{n + 2} & : n \text{ and } \frac{n}{2} \text{ are even} \\
  2x + x^2 + (1 + x)^{n + 2} & : n \text{ is even and } \frac{n}{2} \text{ is odd.}
\end{cases}
\]

**Proof** Let \( D_{2n} \) be dihedral groups of order \( 2n \) and \( \Gamma_{\mathcal{D}_{2n}} \) be its conjugacy class graph.

**Case 1:** \( n \) is odd

From Theorem 2.4, the conjugacy class graph of \( D_{2n} \) is \( K_{n+1} \). By Theorem 2.2 and Proposition 2.4, the clique polynomial of \( \Gamma_{\mathcal{D}_{2n}} \) can be computed as follows:

\[
C\left( \Gamma_{\mathcal{D}_{2n}}^\mathcal{G} ; x \right) = C\left( K_{n+1} ; x \right) + C\left( K_{n+2} ; x \right) - 1 \\
= (1 + x)^{n + 1} + (1 + x)^{n + 2} - 1 \\
= (1 + x)^{n + 1} + (1 + x)^{n + 2}. 
\]

**Case 2:** \( n \) and \( \frac{n}{2} \) are even

From Theorem 2.4, the conjugacy class graph of \( D_{2n} \) is \( K_{\frac{n}{2}} \). By Proposition 2.4, the clique polynomial of \( \Gamma_{\mathcal{D}_{2n}} \) can be obtained as follows:

\[
C\left( \Gamma_{\mathcal{D}_{2n}}^\mathcal{G} ; x \right) = C\left( K_{\frac{n}{2}} ; x \right) \\
= (1 + x)^{\frac{n}{2}}.
\]

**Case 3:** \( n \) is even and \( \frac{n}{2} \) is odd

From Theorem 2.4, the conjugacy class graph of \( D_{2n} \) is \( K_{\frac{n}{2}} \). By Theorem 2.2 and Proposition 2.4, the clique polynomial of \( \Gamma_{\mathcal{D}_{2n}} \) can be computed as follows:

\[
C\left( \Gamma_{\mathcal{D}_{2n}}^\mathcal{G} ; x \right) = C\left( K_{\frac{n}{2}} \cup K_{\frac{n}{2}} ; x \right) \\
= C\left( K_{\frac{n}{2}} ; x \right) + C\left( K_{\frac{n}{2}} ; x \right) - 1 \\
= (1 + x)^{\frac{n}{2}} + (1 + x)^{\frac{n}{2}} - 1 \\
= (1 + x)^{\frac{n}{2}} + 2x + x^2 - 1 \\
= 2x + x^2 + (1 + x)^{\frac{n}{2}}.
\]

**Example 3.2** Let \( G \) be the dihedral group of order 12 \( (n = 6) \), i.e. \( G = D_{12} = \langle a, b : a^6 = b^2 = 1, bab = a^{-1} \rangle \). By Theorem 2.3, the conjugacy classes of \( D_{12} \) are \( \mathcal{C}(e), \mathcal{C}(a) = \{a, a^2\}, \mathcal{C}(a^2) = \{a^2, a^4\} \), \( \mathcal{C}(b) = \{b, a^7b, a^5b\} \) and \( \mathcal{C}(ab) = \{ab, a^5b, a^3b\} \). By Theorem 2.4, if \( \Gamma_{\mathcal{D}_{12}} \) is the conjugacy class graph of \( D_{12} \) with the set of vertices, \( V(\Gamma_{\mathcal{D}_{12}}^\mathcal{G}) = \{\mathcal{C}(a), \mathcal{C}(a^2), \mathcal{C}(b), \mathcal{C}(ab)\} \), then \( \Gamma_{\mathcal{D}_{12}} = K_3 \cup K_2 \). Hence, the clique polynomial of the conjugacy class graph of \( D_{12} \) is

\[
C\left( \Gamma_{\mathcal{D}_{12}}^\mathcal{G} ; x \right) = C\left( K_3 \cup K_2 ; x \right) \\
= 2x + x^2 + (1 + x)^3 \\
= 2x + x^2 + 1 + 2x + x^2 \\
= 1 + 4x + 2x^2.
\]

**CONCLUSION**

In this paper, the independence polynomial and clique polynomial of the conjugacy class graph of the group \( D_{2n} \) are obtained. The results are based on three cases which are when \( n \) is odd, when \( n \) and \( \frac{n}{2} \) are even, and also when \( n \) is even and \( \frac{n}{2} \) is odd. The independence polynomial of the conjugacy class graph of \( D_{2n} \) is

\[
I\left( \Gamma_{\mathcal{D}_{2n}}^\mathcal{G} ; x \right) = 1 + \left( \frac{n + 1}{2} \right)x + \left( \frac{n}{2} \right)x^2 \text{ when } n \text{ is odd,} \\
I\left( \Gamma_{\mathcal{D}_{2n}}^\mathcal{G} ; x \right) = 1 + \left( \frac{n + 2}{2} \right)x \text{ when } n \text{ and } \frac{n}{2} \text{ are even, and} \\
I\left( \Gamma_{\mathcal{D}_{2n}}^\mathcal{G} ; x \right) = 1 + \left( \frac{n}{2} \right)x + \left( \frac{n}{2} \right)x^2 \text{ when } n \text{ is even and } \frac{n}{2} \text{ is odd.}
\]

Meanwhile, the clique polynomial of the conjugacy class graph of \( D_{2n} \) is \( C\left( \Gamma_{\mathcal{D}_{2n}}^\mathcal{G} ; x \right) = x + (1 + x)^{\frac{n}{2}} \) when \( n \) is odd. When \( n \) and \( \frac{n}{2} \) are even, \( C\left( \Gamma_{\mathcal{D}_{2n}}^\mathcal{G} ; x \right) = (1 + x)^{\frac{n}{2}} + 2x + x^2 \) when \( n \) is even and \( \frac{n}{2} \) is odd. \( C\left( \Gamma_{\mathcal{D}_{2n}}^\mathcal{G} ; x \right) = 2x + x^2 + (1 + x)^{\frac{n}{2}} + 2x + x^2 \).

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