Malaysian Journal of Fundamental & Applied Sciences Vol.8, No.1 (2012) 38-43.

Malaysian Journal of Fundamental & Applied Sciences

available online at http://mjfas.ibnusina.utm.my

Numerical Conformal Mapping of Unbounded Multiply Connected Regions onto Circular Slit Regions

A.A.M. Yunus[1], A.H.M. Murid[1,*] & M.M. S. Nasser[2,3]

[1]*Department of Mathematics, Faculty of Science, Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor, Malaysia,*
[2]*Department of Mathematics, Faculty of Science, King Khalid University, P.O. Box 9004, Abha, Saudi Arabia,*
[3]*Department of Mathematics, Faculty of Science, Ibb University, P.O. Box 70270, Ibb, Yemen*

Received 5 November 2011, Revised 2 January 2012, Accepted 7 January 2012, Available online 28 January 2012

ABSTRACT

This paper presents a boundary integral equation method for conformal mapping of unbounded multiply connected regions onto circular slit regions. Three linear boundary integral equations are constructed from a boundary relationship satisfied by an analytic function on an unbounded multiply connected region. The integral equations are uniquely solvable. The kernels involved in these integral equations are the classical and the adjoint generalized Neumann kernels. Several numerical examples are presented.

| Numerical conformal mapping | Boundary integral equations | Unbounded Multiply Connected Region | Neumann kernel | Generalized Neumann kernel |

http://dx.doi.org/10.11113/mjfas.v8n1.120

1. INTRODUCTION

Conformal mapping is a special mapping that uses function of complex variable to transform a planar region onto another planar region while the angles between curves are preserved in magnitude as well as in their direction. With regards to conformal mapping, canonical region is known as a set of finitely connected regions *S* such that each finitely connected non-degenerate region is conformally equivalent to a region in *S.* There are several types of canonical region for multiply connected regions as listed in [1] and [2]. The class of canonical regions includes slits regions and circular regions. For slit regions, there are five types of such regions which are (i) disk with concentric circular slit, (ii) annulus with concentric circular slit, (iii) circular slit regions, (iv) radial slit region, (v) parallel slit region.

One major setback in conformal mapping is that only for certain regions are exact conformal maps known. One way to deal with this limitation is by numerical computation. Trefethen [3] has discussed several methods for computing conformal mapping numerically. Boundary integral equation related to a boundary relationship satisfied by a function which is analytic in a simply connected region interior to a closed smooth Jordan curve has been given by [4] and [5].

**Corresponding author at:*
E-mail address: alihassan@utm.my

Special realizations of this integral equation are the integral equations related to the Szegö kernel, Bergmann kernel and Riemann map. The kernels arise in these integral equations are the Neumann kernel and the Kerzman-Stein kernel.

Hu [6] and Murid and Hu [7] managed to construct a boundary integral equation for numerical conformal mapping of bounded multiply connected region onto a unit disk with slits. However, the integral equation involves unknown radii which lead to a system of nonlinear equation after the discretization of the integral equation. Nasser [8] produces another technique for numerical conformal mapping of multiply connected regions by expressing the mapping function in terms of the solution of a uniquely solvable Riemann-Hilbert problem. This uniquely solvable Riemann Hilbert problem can be solved by means of boundary integral equation with the generalized Neumann kernel. Sangawi et al. [9] have constructed new linear boundary integral equations for conformal mapping of bounded multiply region onto a unit disk with circular slits, which improves the work of [7] and [4]. Recently, Yunus et al. [10] managed to extent work by [4] and [9] for numerical conformal mapping of unbounded multiply connected region onto exterior unit disk with circular slits.

In this paper we construct some integral equations for numerical conformal mapping of unbounded multiply connected regions onto the circular slit region. The boundary integral equations are constructed from a boundary relationship satisfied by an analytic function on an unbounded multiply connected region.

2. Notation and Auxiliary Material

Suppose Ω^- denotes the unbounded multiply connected region of connectivity *M*. The boundary Γ consist of M smooth Jordan curves Γ_j, *j*= 1, 2, 3…, *M* and shall be denoted by $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_M$. The boundaries are assumed in clockwise direction (see Figure 1). Each curve Γ_k is parameterized by 2π-periodic twice continuously differentiable complex function $z_k(t)$ with non-vanishing first derivative

$$z'_k(t) = \frac{dz_k(t)}{dt} \neq 0, \quad t \in J_k = [0, 2\pi], \qquad k = 1, ..., N.$$

The total parameter J is the disjoint union of M intervals, $J_k \in [0, 2\pi], k = 1, 2, ..., M$. We define a parameterization $z(t)$ of the whole boundary Γ on J by

$$z(t) = \begin{cases} z_1(t), & t \in J_1 = [0, 2\pi], \\ \vdots & \\ z_M(t), & t \in J_M = [0, 2\pi]. \end{cases}$$

Let $f(z)$ be the conformal mapping function that maps Ω^- onto U^-, where U^- represents the circular slit region. The boundary correspondence function $\theta(t)$ for $f(z)$ can be written as

$$\theta(t) = \begin{cases} \theta_1(t), & t \in J_1 = [0, 2\pi], \\ \vdots & \\ \theta_M(t), & t \in J_M = [0, 2\pi]. \end{cases}$$

The radii (piecewise real constant function) of the radial slits for $t \in J$ can be represented by

$$\mu(t) = \begin{cases} \mu_1 & t \in J_1 = [0, 2\pi], \\ \vdots & \\ \mu_M & t \in J_M = [0, 2\pi]. \end{cases}$$

For simplicity, the piecewise constant function R will be denoted as $\mu = (\mu_1, \mu_2, ..., \mu_M)$. Let $A(t)$ be a complex continuously differentiable 2π-periodic function for all $t \in J$. We define the generalized Neumann kernel formed with A as [11]

$$N(t,s) = \frac{1}{\pi} \mathrm{Im}\left(\frac{A(t)}{A(s)} \frac{z'(s)}{z(s) - z(t)} \right).$$

When *A*=1, the generalized Neumann kernel reduces to the classical Neumann kernel, i.e.

$$N(t,s) = \frac{1}{\pi} \mathrm{Im}\left(\frac{z'(s)}{z(s) - z(t)} \right).$$

The adjoint kernel *N**(t,*s*) of the classical Neumann kernel is given by

$$N^*(t,s) = N(s,t) = \frac{1}{\pi} \mathrm{Im}\left(\frac{z'(t)}{z(t) - z(s)} \right).$$

We define the Fredholm integral operators **N** and **N*** by

$$\mathbf{N}\upsilon(t) = \int_J N(t,s)\upsilon(s)ds, \qquad t \in J,$$

$$\mathbf{N^*}\upsilon(t) = \int_J N^*(t,s)\upsilon(s)ds, \qquad t \in J.$$

The eigenfunctions of *N*(*t*,*s*) corresponding to the eigenvalue $\lambda = -1$ are $\{\chi^{[1]}, \chi^{[2]}, \ldots, \chi^{[M]}\}$, where [11]

$$\chi^{[j]}(\xi) = \begin{cases} 1, & \xi \in \Gamma_j, \\ 0, & \text{otherwise.} \end{cases}$$

Lastly, we define an integral operator **J** as [9]

$$\mathbf{J}v = \left(\frac{1}{2\pi} \int_{J_1} v_1(s)ds, \frac{1}{2\pi} \int_{J_2} v_2(s)ds, \ldots, \frac{1}{2\pi} \int_{J_M} v_M(s)ds \right)$$

which is required for uniqueness of the solution in Section 4 and to find the values of the constants.

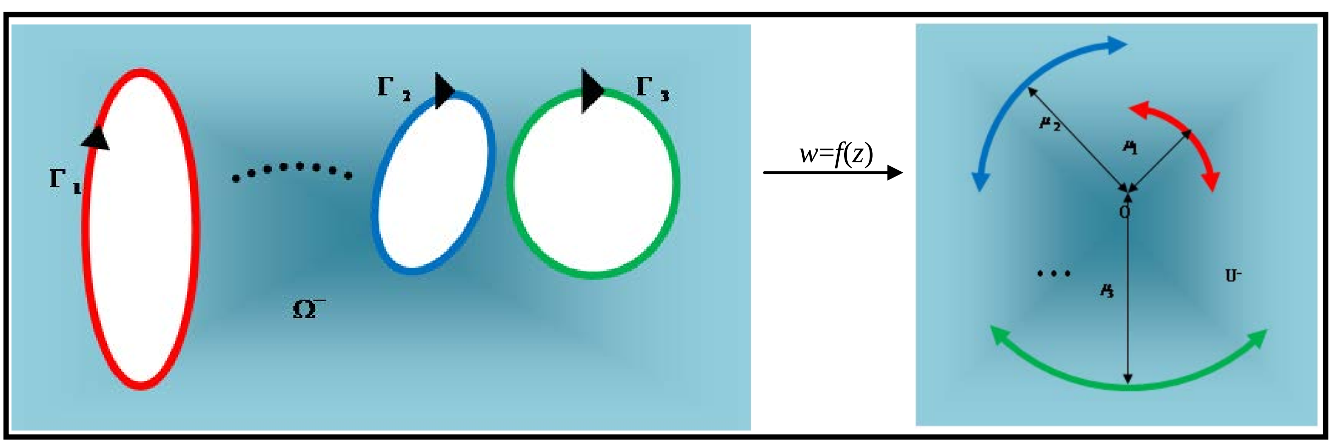

Figure 1. Mapping of Ω^- onto U^-

3. Non-Homogeneous Boundary Relationship for Conformal Mapping of Unbounded Region

Suppose we are given a function $D(z)$ which is analytic with respect to $z \in \Omega^-$, continuous on $\Omega^- \cup \Gamma$, Hölder continuous on Γ and $D(\infty)$ is finite. The boundary Γ_j is assumed to be a smooth Jordan curve. The unit tangent to Γ at the point $z \in \Gamma$ will be denoted by $T(z)$. Suppose further that $D(z)$ satisfies the exterior non-homogeneous boundary relationship

$$D(z) = c(z)\left[\frac{T(z)Q(z)D(z)}{P(z)}\right]^{\text{conj}} + \overline{H(z)}, \qquad z \in \Gamma. \qquad (1)$$

where the symbol "conj" denotes complex conjugation, $c(z)$, $H(z)$, $Q(z)$ and $P(z)$ are complex-valued functions defined on Γ with the following properties.

- $P(z)$ is analytic with respect to $z \in \Omega^-$, continuous on $\Omega^- \cup \Gamma$,
- $P(\infty) \neq 0, D(\infty)$ is finite,
- $c(z) \neq 0, P(z) \neq 0$ and $Q(z) \neq 0$ for $z \in \Gamma$,
- $H(z)/Q(z)T(z) \in \text{Lip}\lambda.$

Note that the boundary relationship (1) also has the following equivalent form:

$$P(z) = \overline{c(z)}T(z)Q(z)\frac{D(z)^2}{|D(z)|^2} + \frac{P(z)H(z)}{D(z)}, \qquad z \in \Gamma. \qquad (2)$$

Under these assumptions, an integral equation for D can be constructed by means of the following theorem:

Theorem 1. *Let u and v be complex-valued functions defined on* Γ. *Then*

$$\begin{aligned}
&\frac{1}{2}\left[v(z) + \frac{\overline{u(z)}}{\overline{T(z)Q(z)}}\right]D(z) \\
&\quad + PV\frac{1}{2\pi i}\int_\Gamma \left[\frac{c(z)u(z)}{c(w)(\overline{w}-\overline{z})\overline{Q(w)}} - \frac{v(z)T(w)}{w-z}\right]D(w)|dw| \\
&\quad = c(z)u(z)\frac{\overline{D(\infty)}}{\overline{P(\infty)}} + v(z)D(\infty) \\
&\quad - c(z)u(z)\sum_{j=1}^{m}\left[\operatorname*{Res}_{w=a_j}\frac{D(w)}{(w-z)P(w)}\right]^{\text{conj}} - u(z)\overline{L^+(z)}. \quad z \in \Gamma \qquad (3)
\end{aligned}$$

where

$$L^+(z) = -\frac{1}{2}\frac{H(z)}{Q(z)T(z)} + \text{PV}\frac{1}{2\pi i}\int_\Gamma \frac{\overline{c(z)}H(w)}{\overline{c(w)}(w-z)T(w)Q(w)}dw.$$

The sum in the Theorem 1 is over all those zeros lies inside Ω^-. However, if $P(z)$ does not have zeros, the term containing residue in Theorem 1 will not be appear. If $H(z)$=0 then (1) is known as homogeneous boundary relationship and the term which contain $H(z)$ above will not appear. The symbol L^+ denotes boundary values from outside Ω^-.

4. Application of Boundary Integral Equation for Conformal Mapping of Unbounded Multiply Connected Region onto Circular Slits Regions

The canonical region U^- consist of M slits along the circle $|w| = \mu_k$, where k=1,2,...,M and μ_k is undetermined real constant. Let $w = f(z)$ be the analytic function that maps conformally Ω^- onto U^-. The boundary value of $f(z)$ can be represented in the form

$$f(z_k(t)) = \mu_k e^{i\theta_k(t)}, \quad z = z_k(t), \quad 0 \le t \le \beta_k \quad (4)$$

where $\theta_p(t)$ denotes boundary correspondence function of Γ_p and μ_p is the radius of a circular slit. Thus, it can be shown that (4) can be rewritten as

$$f(z_p(t)) = \text{sign}(\theta'_p(t))\frac{|f(z_p(t))|}{i}T(z_p(t))\frac{f'(z_p(t))}{|f'(z_p(t))|}. \qquad (5)$$

This boundary relationship is useful for computing the boundary values of $f(z)$ provided $\theta'(t)$, $|f(z)|$ and $f'(z)$ are all known. The integral equations for finding all these unknown functions are discussed next.

4.1 Integral Equation Method for Computing μ_p

The mapping function $w = f(z(t))$ can be uniquely determined by assuming $f(\beta) = 0, f(\infty) = \infty$ and $f'(\infty) = 1$. We assume that β is a prescribed point located inside Ω^-. Thus, the mapping function can be expressed as [8]

$$f(z) = (z-\beta)e^{h(z)}, \qquad (6)$$

where $h(z)$ is analytic in Ω^- with $h(\infty) = 0$. By taking logarithm on both sides of (6), we obtain

$$\log f(z(t)) = h(z(t)) + \log(z(t) - \beta). \qquad (7)$$

Further arrangements yields,

$$h(z(t)) = \gamma(t) + r(t) + i\Phi \qquad (8)$$

where ,

$$r(t) = \ln|f(z(t))|,$$
$$\gamma(t) = -\ln|z(t) - \beta|,$$
$$\Phi(t) = \theta(t) - \arg(z(t) - \beta).$$

The value of $r(t)$ and $\Phi(t)$ are both unknown. However, the values for $r(t)$ can be calculated by using the following theorem as given in [12].

THEOREM 2. *The function r is given $r=(r_1,r_2,\ldots,r_M)$, where*

$$r_j = (\gamma, \phi^{[j]}) = \frac{1}{2\pi}\int_{\Gamma} \gamma(t)\phi^{[j]}(t)dt,$$

and $\phi^{[j]}$ is the unique solution of the following integral equation

$$(\mathbf{I}+\mathbf{N}^{*}+\mathbf{J})\phi^{[j]} = -\chi^{[j]}, \qquad j = 1,2,\ldots,N.$$

By obtaining $r_1,r_2,\ldots,r_M$, the value for the unknown radii $\mu_q, q = 2,3,\ldots,M$, can be obtained by

$$\mu_q = e^{r_q}. \tag{9}$$

4.2 Integral Equation Method for Computing $f'(z)$

By squaring both sides of equation (5) and divide both sides of the equation with $(z(t)-\beta)^2$, we obtain

$$\frac{f(z_p(t))^2}{(z(t)-\beta)^2} = -\frac{\left|f_p(z(t))\right|^2}{(z(t)-\beta)^2}T(z(t))^2\left[\frac{f'(z(t))}{|f'(z(t))|}\right]^2 . \tag{10}$$

Upon comparing (10) and boundary relationship (2) leads to a choice of

$$P(z) = \frac{f(z)^2}{(z(t)-\beta)^2}, P(\infty)=1, Q(z)=T(z), u(z)=\overline{T(z)}^2,$$

$$D(z) = f'(z), c(z) = -\frac{|f(z)|^2}{(z(t)-\beta)^2}, v(z)=1.$$

By means of Theorem 1 yields

$$g(z)+\int_{\Gamma} N^{-}(z,w)g(w)|d(w)| = -\frac{\left|f(z)\right|^2}{(z(t)-\beta)^2}\overline{z'(t)}+z'(t), \tag{11}$$

where

$$g(z) = f'(z)z'(t),$$

$$N^{-}(z,w) = \frac{1}{2\pi i}\left[\frac{z'(t)}{z(t)-w(s)} - \frac{\overline{(w(s)-\beta)^2}}{\overline{(z(t)-\beta)^2}}\frac{|f(z)|^2}{|f(w)|^2}\frac{\overline{z'(t)}}{\overline{(z(t)-w(s))}}\right],$$

$$N^{-}(z,z) = \frac{1}{2\pi i}\operatorname{Im}\frac{z''(t)}{z'(t)} + \frac{1}{\pi i}\frac{\overline{z'(t)}}{\overline{z(t)}}.$$

Numerical evidence suggests that to obtain a unique solution for this region, one need to add M conditions since the eigenvalue -1 of $N^{-}(z,w)$ appears M times. Since $f(z)$ is assumed single-valued, it is also require that the unknown mapping function $f(z)$ satisfies [2] i.e.,

$$\int_{J} g(w)ds = 0. \tag{12}$$

By solving the integral equation (11) with condition (12) will give us a unique condition. However, the kernel $N^{-}(z,w)$ involves unknown parameter μ_p. To overcome this, one need the result of Section 4.1 .

4.3 Integral Equation Method for Computing $\theta'_p(t)$

By taking logarithmic derivatives on (6) yields

$$\frac{f'(z)}{f(z)} = h'(z) + \frac{1}{z-\beta}. \tag{13}$$

Thus,

$$R(z) = \frac{f'(z)}{f(z)} - \frac{1}{z-\beta} = h'(z), \tag{14}$$

is analytic in Ω^{-}. Note that the boundary relationship in (10) can be rewritten as

$$\frac{f'(z(t))}{f(z_p(t))} = -T(z(t))^2\,\overline{\frac{f'(z(t))}{f(z_p(t))}} . \tag{15}$$

By substituting (14) into boundary relationship (15) yields

$$R(z) = -\overline{T(z(t))R(z)} - \frac{\overline{T(z)}}{\overline{z-\beta}} - \frac{1}{z-\beta}, \qquad z_p \in \Gamma_p. \tag{16}$$

Upon comparing (16) and boundary relationship (1) leads to a choice of

$$D(z) = R(z) = \frac{f'(z)}{f(z)} - \frac{1}{z-\beta}, c(z) = -1, Q(z) = T(z),$$

$$P(z) = 1 \text{ and } H(z) = -\frac{T(z)^2}{z-\beta} - \frac{1}{\overline{z}-\overline{\beta}}.$$

Let $u(z) = \overline{T(z)Q(z)}$, and $v(z)=1$. By substituting all these assignments into Theorem 1 and multiply both sides with $T(z)$ yields

$$\frac{f'(z)}{f(z)}T(z) + \frac{1}{2\pi i}\int_{\Gamma}\left[\frac{T(z)}{z-w} - \frac{\overline{T(z)}}{\overline{z}-\overline{w}}\right]T(w)\frac{f'(w)}{f(w)}|dw| = 2i\operatorname{Im}\left[\frac{T(z)}{z-\beta}\right]. \tag{17}$$

Let $z=z(t)$ and $w=w(s)$. Hence, by multiplying both sides of (17) by $|z'(t)|$ and using the fact that $i\theta'(t) = \dfrac{f'(z(t))}{f(z(t))}z'(t)$, integral equation (17) becomes

$$\theta'(t)+\int_{\Gamma} N(s,t)\theta'(s)ds = 2\,\mathrm{Im}\left[\frac{z'(t)}{z(t)-\beta}\right]. \qquad (18)$$

Since $N(s,t)=N^*(t,s)$, the integral equation (18) can be written as an integral equation in operator form

$$(\mathbf{I}+\mathbf{N}^*)\theta' = 2\,\mathrm{Im}\left[\frac{z'(t)}{z(t)-\beta}\right]. \qquad (19)$$

However, $\lambda=-1$ is an eigenvalue of $\mathbf{N}^*$ with multiplicity M. By Theorem 12 in [11], (19) is not uniquely solvable. Note that

$$\mathbf{J}\theta' = \left(\frac{1}{2\pi}\int_{J_1}\theta_1'(s)ds, \frac{1}{2\pi}\int_{J_2}\theta_2'(s)ds, \ldots, \frac{1}{2\pi}\int_{J_N}\theta_3'(s)ds\right),$$

Since for the boundary Γ_k, we have $\theta_k(2\pi)-\theta_k(0)=0$, the function $\theta(t)$ satisfies

$$\mathbf{J}\theta' = (0,0,\ldots,0). \qquad (20)$$

By adding (20) to (19), we obtain

$$(\mathbf{I}+\mathbf{N}^*+\mathbf{J})\theta' = 2\,\mathrm{Im}\left[\frac{z'(t)}{z(t)-\beta}\right]. \qquad (21)$$

Thus the integral equation (21) is uniquely solvable [12].

4.4 Mapping of the exterior points

By solving the integral equations in Section 4.1 will give us the value for unknown parameters μ_q and by using this information, integral equation in Section 4.2 can be solved linearly. The value for $\theta'(t)$ can be obtain by solving integral equation in Section 4.3. With all these information, the boundary value of $f(z)$ is computed by

$$f(z) = \mathrm{sign}(\theta'(t))\frac{|f(z(t))|}{i}\frac{g(z)}{|g(z)|}, \qquad z\in\Gamma\ . \qquad (22)$$

Let $f(z)$ be written as

$$f(z)=(z-\beta)F(z), \qquad (23)$$

where $F(z)=f(z)/(z-\beta)$ is analytic in Ω^- with $F(\infty)=1$. Then for every $z\in\Omega^-$, the value of $F(z)$ can be calculated by Cauchy integral formula

$$F(z)=1+\frac{1}{2\pi i}\int \frac{F(w)}{w-z}dw, \qquad w\in\Gamma,\ z\in\Omega^-. \qquad (24)$$

By using the information in (24), the exterior point of the function $f(z)$ can be calculated by (23).

4.5 Numerical Example

For numerical experiment, we used a test region with connectivity three. All the computations were done by using Matlab R2008a software. The number of collocation point on each boundary component is n=512 points. The test region and the corresponding image are shown in Figure 2.

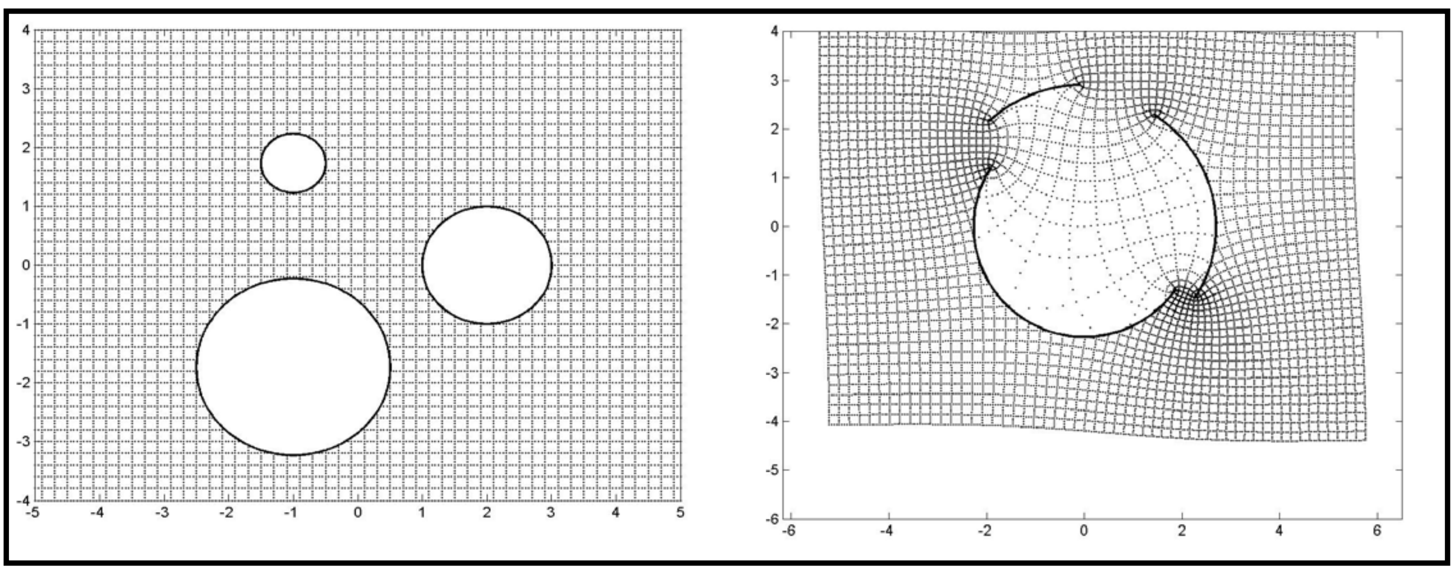

Figure 2. The original region and its image by using our method

5. CONCLUSION

In this paper, we have constructed some new boundary integral equations for numerical conformal mapping of unbounded multiply connected region onto the circular slit regions. The advantage of this method is that the integral equations obtained are linear.

ACKNOWLEDGEMENT

This work was supported in part by the Nanotechnology Research Alliance through Applied Algebra and Analysis Research Group (A^3G). This support is gratefully acknowledged.

REFERENCES

[1] Z. Nehari, Conformal Mapping, Originally published by the McGraw-Hill in 1952 New York: Dover, 1975.
[2] P. Henrici, Applied and Computational Complex Analysis, Vol. 3, John Wiley, New York, 1986.
[3] L. N. Trefethen, Numerical Conformal Mapping. Amsterdam: North-Holland. 1986.
[4] A. H. M. Murid, Boundary Integral Equation Approach for Numerical Conformal Mapping. Ph. D. Thesis. Universiti Teknologi Malaysia, Johor Bahru. 1999.
[5] A. H. M. Murid, and M. R. M. Razali, An Integral Equation Method for Conformal Mapping of Doubly Connected Regions. Matematika. 1999. 15-2:79-93.
[6] L. N. Hu,. Boundary Integral Equations Approach for Numerical Conformal Mapping of Multiply Connected Regions. Ph. D. Thesis. Universiti Teknologi Malaysia, Johor Bahru. 2009
[7] A. H. M. Murid, and L. N. Hu. Numerical Experiment on Conformal Mapping of Doubly Connected Regions onto a Disk with a Slit, International Journal of Pure and Applied Mathematics, 51 (4), 2009, 589-608
[8] M. M. S. Nasser, Numerical Conformal Mapping via a Boundary Integral Equation with the Generalized Neumann Kernel. SIAM J. Sci. Comput. 2009. 31. 1695-1715
[9] A. W. K.. Sangawi, A. H. M. Murid, and M. M. S. Nasser, Linear Integral Equations for Conformal Mapping of Bounded Multiply Connected Regions onto a Disk with Circular Slits. Applied Mathematics and Computation.2011. Vol. 218, 2055-2068.
[10] A. A. M.Yunus, A H.M. Murid, and M. M. S. Nasser,. Boundary Integral Equation Method for Conformal Mapping of Unbounded Multiply Connected Regions onto Exterior Unit Disk with Circular Slits. In Proceeding of Simposium Kebangsaan Sains Matematik Ke-19. UiTM Pulau Pinang, Malaysia, 2011pp. 310-317.
[11] R.Wegmann, and M.M.S, Nasser, The Riemann-Hilbert Problem and the Generalized Neumann Kernel on Multiply Connected Regions. J. Comput. Appl. Math. 2008. 214: 36-57.
[12] M. M. S. Nasser, A.H. M. Murid, M. Ismail, and E.M.A. Alejaily, Boundary Integral Equation with the Generalized Neumann Kernel for Laplace's Equation on Multiply Connected Regions, Applied Mathematics and Computations .2011.217 : 4710-4727.