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Geometrical Representation of Automata over Some Abelian Groups

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ABSTRACT

One of the classic models of automata is *finite automata*, which determine whether a string belongs to a particular language or not. The string accepted by automata is said to be *recognized* by that automata. Another type of automata, so-called *Watson-Crick automata*, with two reading heads that work on double-stranded tapes using the complimentary relation. Finite automata over groups extend the possibilities of finite automata and allow studying the properties of groups using finite automata. In this paper, we consider finite automata over some Abelian groups $\mathbb{Z}_n$ and $\mathbb{Z}_n \times \mathbb{Z}_n$. The relation of Cayley table to finite automata diagram is introduced in the paper. Some properties of groups $\mathbb{Z}_n$ and $\mathbb{Z}_n \times \mathbb{Z}_n$ in terms of automata are also presented in this paper.

| Finite automata | Watson-Crick automata | Group | $\mathbb{Z}_n$ | $\mathbb{Z}_n \times \mathbb{Z}_n$ |

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1. INTRODUCTION

The theory of automata is a mathematical theory which studies abstract computing devices. The abstract computing devices are also called *machines*, which are used to accept an input or string by the transition rules. A symbol or string is said to be *accepted* by an automaton when the symbol or string can be scan or read by the automaton with certain transition rules. There are many types of automata which have been discovered. One type of automata is the *finite automata*, which is first studied by Stephen Kleene in 1950 [1]. The strings accepted by this type of automata will be categorized as a *regular language* [1].

Furthermore, there is another type of automata that is introduced in [2] namely the *Watson-Crick automata*, the automata which are based on the idea of finite automata, and are used to scan the complete DNA molecules. Watson-Crick automata are automata with two reading heads that works on double-stranded tapes using the complimentary relation. One of the main features of these automata is that it can scan two strands of the input on corresponding positions which relates to a complementarity relation similar with the Watson-Crick complementarity of DNA nucleotides. The two strands of the input are then separately scanned from the left to the right by reading only the heads that are controlled by a common state. The result of the upper strand and the lower strand that scanned by Watson-Crick automata is then complementarily related to each other [3]. '

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In group theory, the Cayley table is a square table that is constructed with the operation of addition or multiplication acting on it. It is used to arrange all the possible product of elements of finite groups with operations depending on the group given. In order to study automata with groups, one of the ways is by marking each of the states of automata with certain element of the groups. Thus, the modified automata can now be used to recognize the data given by the Cayley table of the groups. Therefore, the accepted data of each Cayley table can be recognized by an automaton diagram.

From the data given in Cayley table of a group, an automaton diagram can be constructed. Thus, a group can be said to be recognised by automata if an automaton diagram can be constructed from the data given in the Cayley table of the group. Hence, the properties of the group can be analysed by using the automata diagram when the group is recognized by the automata.

Therefore in this paper, the recognition of Cayley table by an automaton is studied modified finite automata are used to recognize the group $\mathbb{Z}_n$ by constructing the automata diagram from the data given in the Cayley table of the group. Some examples of automaton diagram that accept group $\mathbb{Z}_n$ are also presented.

Moreover, the features of these *Watson-Crick automata* of the complimentary relation are then used to recognize the group $\mathbb{Z}_n \times \mathbb{Z}_n$ by the automata diagram. The group $\mathbb{Z}_n \times \mathbb{Z}_n$ is recognized by automata when the data given from Cayley table are accepted by the automata by construction of an automaton diagram.

Finally, some theorems on the recognition of automata over groups $\mathbb{Z}_n$ and $\mathbb{Z}_n \times \mathbb{Z}_n$ are presented in this.

Some of the definitions that will be used throughout this paper are given in the following section.

2. PRELIMINARIES

The basic concepts of automata theory is studied under the field of formal language theory. Below are some definitions and terms used in formal language theory.

Definition 1 [4]: **Alphabet**
A finite, nonempty set V of symbols is called an **alphabet**.

Definition 2 [4]: **String**
Any finite sequence of symbols from an alphabet is called a **string**.

Definition 3 [4]: **Language**
A **language** is a set of all strings of which are chosen from some V^*, where V is a set of particular alphabets.

Definition 4 [4]: **Regular**
A language is called **regular** if and only if there exists some deterministic finite automaton M such that $L = L(M)$.

A model can be expressed by or identified with a language [4]. The empty string which is a string with no symbol is denoted by ε. If A is an alphabet, the symbol A^* is used to denote the set of strings obtained by concatenating zero or more symbols from A. Any subset of A^* is called a language over A.

For the finite automata, the formal definitions of deterministic and nondeterministic finite automata are stated in the following.

Definition 5 [4]: **Deterministic finite automaton**
A **deterministic finite automaton** or DFA is defined by the quintuple

$$M = (Q, \sum, \delta, q_0, F),$$

where,
Q is a finite set of internal states,
$\sum$ is a finite set of symbols called the input alphabet,
δ: Q x $\sum$ is a transition function,
$q_0 \in Q$ is the initial state, and
$F \subseteq Q$ is a set of final states.

Definition 6 [5]: **Non-deterministic finite automaton**
A **nondeterministic finite automaton** or NFA is defined by the quintuple

$$M = (Q, \sum, \delta, q_0, F),$$

where,
Q is a finite set of internal states,
$\sum$ is a finite set of symbols called the input alphabet,
δ is a transition function where $\delta \subseteq \{Q \text{ x } (\sum \cup \{\lambda\} \text{ x } Q)\}$
$q_0 \in Q$ is the initial state, and
$F \subseteq Q$ is a set of final states.

For the automata with two reading heads, namely the Watson-Crick automata, the formal definition is given in the following.

Definition 7 [2] Watson- Crick finite automata
A **Watson-Crick finite automaton** is a 6-tuple

$$M = (V, \rho, K, s_0, F, \delta),$$

where,
V is the alphabet of the automaton,
K is a set of final states,
$\rho \subseteq V \times V$ is a symmetric relation (the complementarity relation),
$s_0 \in K$ is the initial state,
$F \subseteq K$ is the set of final states and
δ: $K \times (V^*, V^*) \rightarrow 2^K$ is a mapping such that $(s, (x, y)) \neq \emptyset$ only for finitely many triples $(s, x, y) \in K \times V^* \times V^*$.

To relate automata theory to groups, some definitions of group theory are given in the following.

Definition 8 [6] **Group**
A **group** G is a nonempty set represented as an ordered pair $(G, *)$, where G is a set and * is a binary operation on G satisfying the following axioms:

i. closure,
ii. associativity,
iii. identity,
iv. inverse.

Definition 9 [6] **Monoid**
An algebraic structure $(M, *)$ is said to be a **monoid** if it satisfies the following properties:

i. closure,
ii. associative,
iii. identity.

In the next section, the relation of Cayley table to an automaton is studied.

3. RELATION OF THE CAYLEY TABLE TO AN AUTOMATON

A Cayley table is constructed by rows and columns which are labeled by the representatives. The entry in row x and column y is the unique representative z such that $z \equiv xy$.

The concept of constructing a diagram for an automaton from the given information in a Cayley table is similar to the concept of constructing a transition table from an automaton. This method of constructing a transition table from an automaton is introduced in [1]. Now, an automaton can be constructed from any given transition table.

With the information given in the transition table, an automaton diagram can be constructed. For example, given a transition table of automaton A (table 1).

Table 1: Transition table of an automaton *A*

*	1	2
1	2	1
2	2	3
3	3	3

This table can be accepted by a transition graph as shown in figure 1.

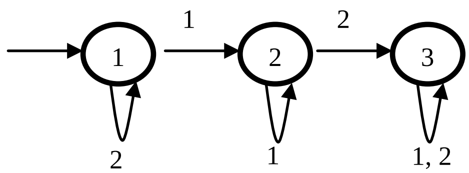

Figure 1: Transition graph of an automaton *A*

With the same concept of constructing the diagram for an automaton from a given transition table, the information in a Cayley table can be represented by a labeled directed graph called the Cayley graph of automaton. The Cayley graph of automaton is constructed using some vertices and some arrows which are pointed from one vertex to another by labeled edge e_{ij}. That is, for the representation $y \equiv xa$, for some vertex x and $a \in A$, then some arrows pointing from x into y by labeled edge as a can be drawn [1].

By the information given in the Cayley table (Table 2), a Cayley graph of an automaton can be constructed.

Table 2: Cayley table of an automaton *B*

	ε	a	b
ε	ε	a	b
a	a	a	a
b	b	b	b

Figure 2 shows an example of Cayley graph of an automaton using the information given in Table 2.

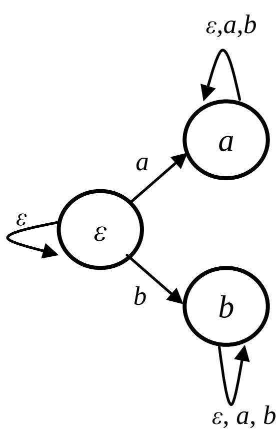

Figure 2: Cayley graph of an automaton *B*

Next, the recognition of the automaton for groups $\mathbb{Z}_n$ and $\mathbb{Z}_n \times \mathbb{Z}_n$ are discussed in the following section.

4. AUTOMATA DIAGRAM FOR SOME ABELIAN GROUPS $\mathbb{Z}_n$ AND $\mathbb{Z}_n$ x $\mathbb{Z}_n$

In the study of automata over groups, one can determine whether the groups can be recognized by some automata. Now, consider the group ($\mathbb{Z}_n$, +, 0). It can be concluded that ($\mathbb{Z}_n$, +, 0) is an abelian group where 0 is the identity element of the group. Thus, a Cayley table for the group ($\mathbb{Z}_n$, +, 0) can be obtained. From the table, one can determine whether the group can be accepted by finite automaton. A group can be accepted by an automaton if a Cayley graph can be drawn from the Cayley table of the group. Therefore, an automaton diagram can be constructed from the given Cayley table.

To recognize the data given in the Cayley table of groups ($\mathbb{Z}_n$, +, 0), modified deterministic finite automata is used, as mentioned in Definition 10. The definition of the *Modified finite automata* is given in Definition 10.

Definition 10: Modified deterministic finite automaton
For a group K of ($\mathbb{Z}_n$, 0, +). A *modified deterministic finite automaton* is defined as

$$M = (Q, \sum, K, \delta, q_0, F),$$

where,
Q is a finite set of internal states,
$\sum$ is a finite set of symbols called the input alphabet,
$q_0 \in Q$ is the initial state,
$F \subseteq Q$ is a set of final states.
such that $Q, \sum \subseteq K$, with transition function,

$$\delta: Q * \sum \longrightarrow Q,$$

and binary operation $*$ defined as $q * b = (q + b) \bmod n$ for $q \in Q$, $b \in \sum$.

The groups ($\mathbb{Z}_2$, +, 0), ($\mathbb{Z}_3$, +, 0) and ($\mathbb{Z}_4$, +, 0) that are recognized by an automaton are shown in Example 1, 2 and 3 respectively.

Example 1:

For the group $\mathbb{Z}_2 = \{0, 1\}$, the Cayley table for $\mathbb{Z}_2$ is shown in Table 3.

Table 3: Cayley table for $\mathbb{Z}_2$

+	0	1
0	0	1
1	1	2

Then from the data that given in the Cayley table (Table 3), the group $\mathbb{Z}_2$ can be recognized by an automaton diagram as shown in Figure 3.

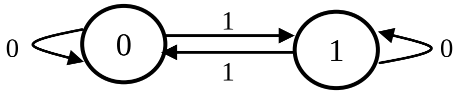

Figure 3: Automaton diagram for $\mathbb{Z}_2$

Example 2:

For the group $\mathbb{Z}_3 = \{0, 1, 2\}$, the Cayley table for $\mathbb{Z}_3$ is shown in Table 4.

Table 4: Cayley table for $\mathbb{Z}_3$

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Then the group $\mathbb{Z}_3$ can be recognized by an automaton diagram as shown in Figure 4.

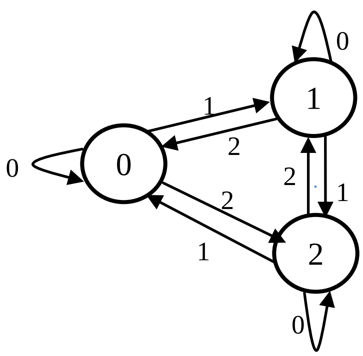

Figure 4: Automaton diagram for $\mathbb{Z}_3$

Next is the example for the group $\mathbb{Z}_4$.

Example 3:

For the group $\mathbb{Z}_4 = \{0, 1, 2, 3\}$, the Cayley table for $\mathbb{Z}_4$ is shown in Table 5.

Table 5: Cayley table for $\mathbb{Z}_4$

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Then the group $\mathbb{Z}_4$ can be recognized by an automaton diagram as shown in Figure 5.

Now, we consider for the case of $\mathbb{Z}_n \times \mathbb{Z}_n$. This group cannot be recognized by *modified deterministic finite automata*. Therefore, in order for the group $\mathbb{Z}_n \times \mathbb{Z}_n$ to be recognized by automata, the concept of *Watson-Crick finite automata* is used to relate the group $\mathbb{Z}_n \times \mathbb{Z}_n$ to automata theory. The complimentarily relation in *Watson-Crick finite automata* plays an important role in order to recognize the direct product of the abelian group $\mathbb{Z}_n$. ie, $\mathbb{Z}_n \times \mathbb{Z}_n$.

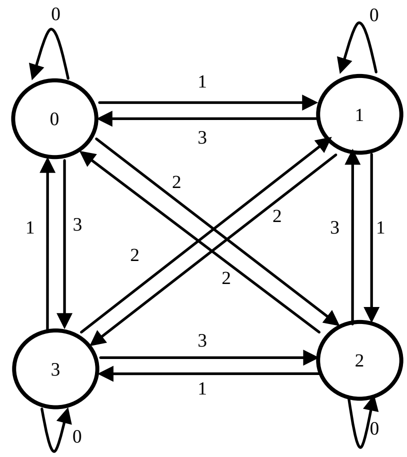

Figure 5: Automaton diagram for $\mathbb{Z}_4$

Similarly, in order to recognize the group $\mathbb{Z}_n \times \mathbb{Z}_n$, an automaton diagram must be constructed from the data given in the Cayley table of group $\mathbb{Z}_n \times \mathbb{Z}_n$. To recognize the data from Cayley table of group $(\mathbb{Z}_n \times \mathbb{Z}_n, 1, 0)$, the automata that used are called *modified Watson-Crick finite automata*. Following is the definition of *modified Watson-Crick finite automata* which is used to recognize the group $\mathbb{Z}_n \times \mathbb{Z}_n$.

Definition 11: Modified Watson-Crick finite automata
For a group K of $(\mathbb{Z}_n \times \mathbb{Z}_n, (0, 0), +)$. A *modified Watson-Crick finite automaton* is defined as 7-tuple

$$M = (\sum, \rho, Q, K, s_0, F, \delta),$$

where,
$\sum$ is the alphabet of the automaton,
Q is a set of final states,
$\rho \subseteq \sum \times \sum$ is a symmetric relation (the complementarity relation),
$s_0 \in Q$ is the initial state,
$F \subseteq Q$ is the set of final states,
such that ρ, $Q \subseteq K$, with transition function

$$\delta: Q * (\sum^*, \sum^*) \rightarrow Q$$

is a mapping such that $(s, (x, y)) \neq \emptyset$ only for finitely many triples $(s, x, y) \in Q * (\sum^* \times \sum^*)$. The binary operation "$*$" defined as $A * B = ((a_1 + b_1) \bmod n, (a_2 + b_2) \bmod n)$ for $(a_1, a_2) \in A$, $(b_1, b_2) \in B$, $A \in Q$, and $B \in \rho$.

Hence, the data given in the Cayley table for group $(\mathbb{Z}_n \times \mathbb{Z}_n, (0, 0), +)$ can be recognized by an automaton diagram by using the modified Watson-Crick automata. The example for group $((\mathbb{Z}_2 \times \mathbb{Z}_2), (0, 0), +)$ recognized by the modified Watson-Crick automata can represented as an automation diagram as shown in Example 4.

Example 4:

Given a group $((\mathbb{Z}_2 \times \mathbb{Z}_2), (0, 0), +)$ where $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{0, 1\} \times \{0, 1\}$. Then, $(\mathbb{Z}_2 \times \mathbb{Z}_2) = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. Hence, the Cayley table for $\mathbb{Z}_2 \times \mathbb{Z}_2$ is shown in Table 6.

Table 6: Cayley table for $\mathbb{Z}_2 \times \mathbb{Z}_2$

+	(0,0)	(1,0)	(0,1)	(1,1)
(0,0)	(0,0)	(1,0)	(0,1)	(1,1)
(1,0)	(1,0)	(0,0)	(1,1)	(0,1)
(0,1)	(0,1)	(1,1)	(0,0)	(1,0)
(1,1)	(1,1)	(0,1)	(1,0)	(0,0)

Thus, an automaton diagram can be drawn from Table 6 as shown in Figure 6.

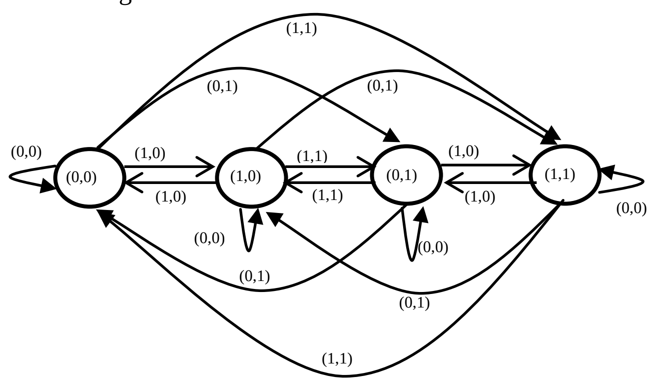

Figure 6: Automaton diagram for $\mathbb{Z}_2 \times \mathbb{Z}_2$

6. SOME PROPERTIES OF ABELIAN GROUPS $\mathbb{Z}_n$ AND $\mathbb{Z}_n \times \mathbb{Z}_n$ IN TERMS OF AUTOMATA

After analyzing on the automaton diagram that recognize groups $\mathbb{Z}_n$ and $\mathbb{Z}_n \times \mathbb{Z}_n$ in Section 5, some theorems for the properties of groups in terms of automata are obtained. Lemma 1, Theorem 2 and 3 are obtained from analyzing the automaton diagram recognized by the group $\mathbb{Z}_n$, while Lemma 2, Theorem 3 and 4 are obtained from analyzing the automaton diagram recognized by the group $\mathbb{Z}_n \times \mathbb{Z}_n$.

Lemma 1:
If there exist a self-loop transition for each of the states, then there is an identity element of the abelian group of $(\mathbb{Z}_n, 0, +)$ in the edge of the self-loop transition. i.e., e_{ij} is the identity if $i = j$ for $i, j \in \mathbb{Z}$.

Proof:
Let $e_{ij} = b$ such that $b = \{b_k \mid \delta(a_i, b_k) = a_j$, for all $a_i, a_j \in Q$ and $b_l \in \sum\}$. The transition function $\delta(a_i, b_k) = a_j$ is defined as $\delta(a_i, b_k) = a_i * b_k$ and the binary operation '$*$' is defined as $a_i * b_k = (a_i + b_k) \bmod n$. Thus, $(a_i + b_k) \bmod n = a_j$.

Suppose that $i = j$. So, $a_i = a_j$. Therefore, $(a_i + b) \bmod n = a_i$. Hence, $a_i * b = a_i$. Since a_i and b are element of abelian group, then $a_i * b = b * a_i$. So, $a_i * b = b * a_i = a_i$. Thus, $b = e_{ij}$ is the identity element if $i = j$.

Theorem 1:
If a group $(\mathbb{Z}_n, 0, +)$ can be recognized by a modified deterministic finite automata, there exist a complete graph with a self-loop transition for each state and n states of $(n - 1)$ transition function with $\delta(a_i, b_k) = a_j$ such that $a_i \neq a_j$ for $a_i, a_j \in Q$, $b_k \in \sum$ for $n \geq 1$ and $i, j, k \in \mathbb{Z}$.

Proof:
By induction, for $n = 1$, we have a group of $(\mathbb{Z}_1, 0, +)$. Thus, $\mathbb{Z}_1$ has only one element. $\mathbb{Z}_1 = \{0\}$, where 0 is the identity element of group $\mathbb{Z}_1$. Then from Lemma 1, it has a transition function, δ such that $\delta(0, b) = 0$, where b is the element on the edge of self-loop transition. Therefore, there exist a self-loop transition for the only state and there also exist $(1 - 1)$ number of transition function $\delta(a_i, b_l) = a_j$ such that $a_i \neq a_j$ for $a_i, a_j \in Q$ and $b_l \in \sum$. Hence, the statement is true for the case when $n = 1$.

Next, by assuming $n = k$ is true. Thus, the group of $\mathbb{Z}_k$ has k number of elements and k number of states with $(k - 1)$ number of transition function $\delta(a_i, b_l) = a_j$ such that $a_i \neq a_j$ for $a_i, a_j \in Q$ and $b_l \in \sum$. Hence, $\mathbb{Z}_k$ will have a total number of $(k^2 - k)$ transition function, δ, with $\delta(a_i, b_l) = a_j$ such that $a_i \neq a_j$ for $a_i, a_j \in Q$ and $b_l \in \sum$.

Now, let $n = k + 1$. Then $\mathbb{Z}_{k+1}$ should have a total number of $(k + 1)^2 - (k + 1)$ transition function $\delta(a_i, b_l) = a_j$ such that $a_i \neq a_j$ for $a_i, a_j \in Q$, $b_l \in \sum$. That means it will have $(k^2 + k)$ number of transition function $\delta(a_i, b_l) = a_j$ such that $a_i \neq a_j$ for $a_i, a_j \in Q$ and $b_l \in \sum$.

So, for the case $n = k + 1$, $\mathbb{Z}_{k+1}$ has k number of state and one additional state of $(k + 1)$. Thus, $\mathbb{Z}_{k+1}$ has additional of one self-loop transition function at the state of $(k + 1)$ and k numbers of transition from each of k number of states to state of $(k + 1)$. Hence, for case $n = k + 1$, $\mathbb{Z}_{k+1}$ has $(k^2 - k) + (k - 1) + (k + 1)$ number of transition $\delta(a_i, b_l) = a_j$ such that $a_i \neq a_j$ for $a_i, a_j \in Q$ and $b_l \in \sum$. That is, $\mathbb{Z}_{k+1}$ has $(k^2 + k)$ number of transition function $\delta(a_i, b_l) = a_j$ such that $a_i \neq a_j$ for $a_i, a_j \in Q$, $b_l \in \sum$.

Therefore, it is also true for the case $n = k + 1$. Therefore, the statement is true for all $n \geq 1$.

Lemma 2:
If there exist a self-loop transition for each of the states, then there is an identity element of the abelian group of $((\mathbb{Z}_n \times \mathbb{Z}_n), (0, 0), +)$ in the edge of the self-loop transition. i.e., e_{IJ} is the identity if $I = J$ for $I, J \in \mathbb{Z}$.

Proof:
From modified Watson-crick automata, the transition function is defined as $\delta: Q * \rho \rightarrow Q$. Suppose that there exist a self-loop transition for each of states, such that $\delta(A_I, e_{IJ}) = A_J$ for $I = J$. Thus, we have $\delta(A_I, e_{II}) = A_I$. Therefore, $\delta(A_I, e_{II}) = A_I * e_{II} = ((a_1 + b_1) \bmod n, (a_2 + b_2) \bmod n)$ for $(a_1, a_2) \in A_I$, $(b_1, b_2) \in e_{II}$, $A_I \in Q$, $e_{II} \in \rho$.
Hence,
$((a_1 + b_1) \bmod n, (a_2 + b_2) \bmod n) = (a_1 \bmod n, a_2 \bmod n)$

So,

$$a_1 \bmod n = (a_1 + b_1) \bmod n \quad (1)$$
$$a_2 \bmod n = (a_2 + b_2) \bmod n \quad (2)$$

From (1) and by properties of modulo arithmetic, we get

$$a_1 \bmod n = (a_1 + b_1) \bmod n = a_1 \bmod n + b_1 \bmod n.$$

Thus,

$$b_1 \bmod n = a_1 \bmod n - a_1 \bmod n = (a_1 - a_1) \bmod n = 0.$$

Similarly for (2), we have

$$a_2 \bmod n = (a_2 + b_2) \bmod n = a_2 \bmod n + b_2 \bmod n$$

Thus,

$$b_2 \bmod n = a_2 \bmod n - a_2 \bmod n = (a_2 - a_2) \bmod n = 0.$$

For $\delta(A_I, e_{IJ}) = A_J$, $A_J * e_{IJ} = A_J$ for $A_I \in Q$, $e_{IJ} \in \rho$. Thus,

$$e_{IJ} = (b_1 \bmod n, b_2 \bmod n) = (0, 0)$$

Hence,
$((a_1 + 0) \bmod n, (a_2 + 0) \bmod n) = (a_3 \bmod n, a_4 \bmod n)$ for $(a_1, a_2) \in A_I$, $(a_3, a_4) \in A_J$, $(b_1, b_2) \in e_{II}$, $A_I, A_J \in Q$, $e_{IJ} \in \rho$. Then, $(a_1 \bmod n, a_2 \bmod n) = (a_3 \bmod n, a_4 \bmod n)$, implies that, $A_I = A_J$. Also, the A_I and e_{IJ} are the element of abelian group $(\mathbb{Z}_n \times \mathbb{Z}_n)$. Therefore,

$$A_I * e_{II} = e_{II} * A_I = A_I.$$

Hence, $e_{IJ} = (b_1, b_2)$ is the identity element of $(\mathbb{Z}_n \times \mathbb{Z}_n)$ if $I = J$.

Theorem 2:
If a group of $((\mathbb{Z}_n \times \mathbb{Z}_n), (0, 0), +)$ can be recognized by modified Waston-Crick finite automata, there exist a complete graph of one self-loop transition for each state and n^2 number of state of $(n^2 - 1)$ transition function with $\delta(a_i, b_k) = a_j$ such that $a_i \neq a_j$ for $a_i, a_j \in Q$, $b_k \in \rho$ for $n \geq 1$ and and $i, j, k \in \mathbb{Z}$.

Proof:
By induction, for $n = 1$, group $((\mathbb{Z}_1 \times \mathbb{Z}_1), (0, 0), +) = \{(0, 0)\}$. Hence by Lemma 2, there exist a self-loop transition such as $\delta((0, 0), (0, 0)) = (0, 0)$ and $(1 - 1)$ number of transition with $\delta(a_i, b_k) = a_j$ such that $a_i \neq a_j$ for $a_i, a_j \in Q$, $b_k \in \rho$. Therefore, the statement is true for $n = 1$.

By assuming that $n = k$ is true. Thus $((\mathbb{Z}_k \times \mathbb{Z}_k), (0, 0), +) = \{(0, 0), (0, 1), \ldots, (0, k), (1, 0), \ldots, (k, k)\}$. So, $(\mathbb{Z}_k \times \mathbb{Z}_k)$ has a total number of $k \times k$ states. From the states, the following transition can be obtained:

$$
\begin{aligned}
(0, 0) &\to (0, 0)\\
&\to (0, 1)\\
&\;\;\vdots\\
&\to (k, k)\\
(1, 0) &\to (0, 0)\\
&\to (0, 1)\\
&\;\;\vdots\\
&\to (k, k)\\
(k, k) &\to (0, 0)\\
&\to (0, 1)\\
&\;\;\vdots\\
&\to (k, k).
\end{aligned}
$$

Therefore, $(\mathbb{Z}_k \times \mathbb{Z}_k)$ has a total number of k^4 transition functions.

By Lemma 2, there exists a self-loop transition for each of the states. Thus, each states have $(k^2 - 1)$ transition function with $\delta(a_i, b_k) = a_j$ such that $a_i \neq a_j$ for $a_i, a_j \in Q$, $b_k \in \rho$. Therefore, there is a total number of $(k^4 - k^2)$ transition function with $\delta(a_i, b_k) = a_j$ such that $a_i \neq a_j$ for $a_i, a_j \in Q$, $b_k \in \rho$.

Suppose for the case $n = k + 1$, $(\mathbb{Z}_{k+1} \times \mathbb{Z}_{k+1})$ should have a total number of $(k + 1)^2 - (k + 1)$ transition function with $\delta(a_i, b_k) = a_j$ such that $a_i \neq a_j$ for $a_i, a_j \in Q$, $b_k \in \rho$. That is,

$$
\begin{aligned}
&(k + 1)^4 - (k + 1)^2\\
&= ((k + 1)(k + 1))^2 - (k + 1)^2\\
&= (k^2 + 2k + 1)(k^2 + 2k + 1) - (k^2 + 2k + 1)\\
&= k^4 + 4k^3 + 5k^2 + 2k.
\end{aligned}
$$

Hence it should have $k^4 + 4k^3 + 5k^2 + 2k$ number of transition function with $\delta(a_i, b_k) = a_j$ such that $a_i \neq a_j$ for $a_i, a_j \in Q$, $b_k \in \rho$.

Now, for the case $n = k + 1$, $(\mathbb{Z}_{k+1} \times \mathbb{Z}_{k+1})$ has k^2 numbers of states and an addition of $(((k + 1)(k + 1)) - k^2)$ number of states. Since there is an addition of $(k^2)((k + 1)(k + 1) - k^2)$ number of transition function in k^2 numbers of states and there is $((k + 1)(k + 1))((k + 1)(k + 1) - k^2))$ number of transition function for $(((k + 1)(k + 1)) - k^2)$ number of states. Therefore, it has a total number of $k^4 + k^2 (k + 1)^2 - k^2 + ((k + 1)(k + 1))((k + 1)(k + 1) - k^2))$. That is,

$$
\begin{aligned}
&k^4 + k^2(k^2 + 2k + 1 - k^2) + (k^2 + 2k + 1)(k^2 + 2k + 1 - k^2)\\
&= k^4 + (k^2)(2k + 1) + ((k + 1)(k + 1))(2k + 1)\\
&= k^4 + (2k^3 + k^2) + (k^2 + 2k + 1)(2k + 1)\\
&= k^4 + 4(k^3) + 6k^2 + 4k + 1.
\end{aligned}
$$

By Lemma 2, each of the states has a self-loop transition function. Hence, there exists $(k^2 + 2k + 1)$ number of self-loop transition for case of $((\mathbb{Z}_{k+1} \times \mathbb{Z}_{k+1}), +)$. Thus,

$$
\begin{aligned}
&k^4 + 4(k^3) + 6k^2 + 4k + 1 - (k^2 + 2k + 1)\\
&= k^4 + 4(k^3) + 6k^2 + 4k + 1 - k^2 - 2k - 1)\\
&= k^4 + 4k^3 + 5k^2 + 2k.
\end{aligned}
$$

Therefore, there is a total number of $k^4 + 4k^3 + 5k^2 + 2k$ transition function with $\delta(a_i, b_k) = a_j$ such that $a_i \neq a_j$ for $a_i, a_j \in Q$, $b_k \in \rho$ for the case when $n = k + 1$. That is same as predicted.

Hence, the statement is true for $n \geq 1$ and $i, j, k \in \mathbb{Z}$.

7. CONCLUSION

In this paper, the concept of relation of Cayley table to automata is studied. Groups are said to be accepted by an automata if an automaton diagram can be constructed by the data given in the Cayley table of groups. Here, the groups $\mathbb{Z}_n$ and $\mathbb{Z}_n$ x $\mathbb{Z}_n$ are accepted by the *modified deterministic finite automata* and *modified Watson-Crick finite automata* respectively. Furthermore, some examples on the finite automaton diagram which can be used to recognize groups $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$ and $\mathbb{Z}_2$ x $\mathbb{Z}_2$ are presented. Some theorems for some properties of groups $\mathbb{Z}_n$ and $\mathbb{Z}_n$ x $\mathbb{Z}_n$ in terms of automata are also given in this paper together with their proofs.

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