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Integral Equation with the Generalized Neumann Kernel for Computing Green's function on Simply Connected Regions

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ABSTRACT

This research is about computing the Green's functions on simply connected regions by using the method of boundary integral equation. The method depends on solving a Dirichlet problem using a uniquely solvable Fredholm integral equation on the boundary of the region. The kernel of this integral equation is the generalized Neumann kernel. The numerical method for solving this integral equation is the Nyström method with trapezoidal rule which leads to a system of linear equations. The linear system is then solved by the Gaussian elimination method. Mathematica plot of Green's function for a test region is also presented.

| Green's function | Dirichlet problem | Integral equation | Generalized Neumann kernel |

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1. INTRODUCTION

The Green's function is considered as one of the most important functions used to solve inhomogeneous differential equations subject to specific initial conditions or boundary conditions. It is also used in physics, specifically in quantum field theory, electrodynamics and statistical field theory.

Several methods have been studied for computing Green's function, such as the conformal mapping method and boundary integral equation methods. Embree and Trefethen (1999) have proposed a new method for the computation of the Green's function in the complex plane corresponding to a set of K symmetrically placed polygons along the real axis. An important special case is a set of K real intervals. The method is based on a Schwarz–Christoffel conformal map of the part of the upper half-plane exterior to the problem domain onto a semi-infinite strip whose end contains $K-1$ slits. From the Green's function one can obtain a great deal of information about polynomial approximations, with applications in digital filters and matrix iterations. By making the end of the strip jagged, the method can be generalized to weighted Green's functions and weighted approximations.

Crowdy and Marshall (2007) presented an analytical formula for the first-type Green's function for Laplace's equation in multiply connected circular domains.

The method is constructive and relies on the use of a special function known as the Schottky–Klein prime function associated with simply and multiply connected circular domains.

Another approach of computing Green's function is through solving a related Dirichlet problem. Henrici (1986) shows a method for solving the Dirichlet problem that does not require any mapping function. The method reformulates the Dirichlet problem as a Fredholm integral equation. Nasser (2007) has proposed a new method for solving the interior and exterior Dirichlet problem in simply connected regions with smooth boundaries. His method is based on uniquely solvable Fredholm integral equation of the second kind with the generalized Neumann kernel. This paper aims at applying Nasser's method to compute the Green's function on simply connected region.

2. AUXILIARY MATERIALS

Let Ω be a bounded simply connected region in the complex plane as shown in Figure 2.1. The boundary of Ω is denoted by Γ which has positive orientation, i.e., in a clockwise direction

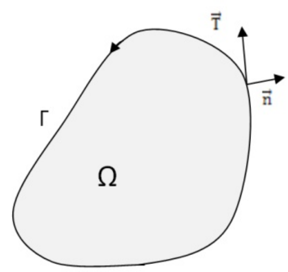

Fig. 2.1 Bounded simply connected region.

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We assume that Γ has a parameterization $\eta(t)$, $t \in I = [0,2\pi]$, which is a complex-valued periodic function with period 2π. The parameterization η also need to be twice continuously differentiable such that

$$\dot{\eta}(t) = \frac{d\eta(t)}{dt} \neq 0. \qquad (2.1)$$

The unit tangent vector to the whole boundary Γ is denoted by

$$T = \frac{\dot{\eta}(t)}{|\dot{\eta}(t)|},$$

while the outward normal direction is defined as

$$n = \frac{-i\,\dot{\eta}(t)}{|\dot{\eta}(t)|}.$$

Let u be a real function defined in the domain Ω and let $x = x + iy \in \Omega$ .In our research, for simplicity, we shall write $u(z)$ instead of $u(x, y)$. Let H^{α} be the space of all real Hölder continuous function with exponent α on the boundary Γ. The interior and the exterior Dirichlet problems are defined as follows.

Interior Dirichlet problem:
Let $\gamma \in H^{\alpha}$ be a given function. Find the function u harmonic in Ω, Holder continuous on Γ and satisfies the boundary condition

$$u(\eta(t)) = \gamma(t), \qquad \eta(t) \in \Gamma. \qquad (2.2)$$

The interior Dirichlet problem (2.2) is uniquely solvable (Atkinson, 1997).

Exterior Dirichlet Problem:
Let $\gamma \in H^{\alpha}$ be a given function. Find the function u harmonic in Ω^{-}, Holder continuous on Γ, $u(z)$ bounded when $|z| \to \infty$ and satisfies the boundary condition

$$u(\eta(t)) = \gamma(t), \qquad \eta(t) \in \Gamma. \qquad (2.3)$$

The exterior Dirichlet problem (2.3) is uniquely solvable (Atkinson, 1997).

Let $A(t)$ be a continuously differentiable 2π-periodic function with $A \neq 0$. We define two real functions N and M by (Wegmann et al. (2005), Nasser (2007))

$$N(\tau,t) = \frac{1}{\pi}\operatorname{Im}\left(\frac{A(\tau)}{A(t)}\frac{\dot{\eta}(t)}{\eta(t)-\eta(\tau)}\right), \qquad (2.4)$$

$$M(\tau,t) = \frac{1}{\pi}\operatorname{Re}\left(\frac{A(\tau)}{A(t)}\frac{\dot{\eta}(t)}{\eta(t)-\eta(\tau)}\right), \qquad (2.5)$$

The kernel $N(\tau,t)$ is called the *generalized Neumann kernel* formed with A and η (Wegmann *et al.*, 2005). When A = 1, the kernel N is the Neumann kernel which arise frequently in the integral equations for potential theory and conformal mapping (Henrici, 1986).

Theorem 2.1 (Wegmann *et al.*, 2005)

a) *The kernel $N(\tau,t)$ is continuous with*

$$N(t,t) = \frac{1}{\pi}\operatorname{Im}\left(\frac{1}{2}\frac{\ddot{\eta}(t)}{\dot{\eta}(t)} - \frac{\dot{A}(t)}{A(t)}\right), \qquad (2.6)$$

b) *The kernel $M(\tau,t)$ has the representation*

$$M(\tau,t) = -\frac{1}{2\pi}\cot\frac{\tau-t}{2} + M_1(\tau,t), \qquad (2.7)$$

with a continuous kernel M_1 which takes on the diagonal the values

$$M_1(t,t) = \frac{1}{\pi}\operatorname{Re}\left(\frac{1}{2}\frac{\ddot{\eta}(t)}{\dot{\eta}(t)} - \frac{\dot{A}(t)}{A(t)}\right), \qquad (2.8)$$

Let N and M_1 be the Fredholm integral operators associate with the continuous kernels N and M_1, i.e.,

$$(N\mu)(\tau) = \int_0^{2\pi} N(\tau,t)\mu(t)dt, \qquad (2.9)$$

$$(M_1\mu)(\tau) = \int_0^{2\pi} M_1(\tau,t)\mu(t)dt, \qquad (2.10)$$

Let also M and K be the singular integral operators

$$(M\mu)(\tau) = \int_0^{2\pi} M(\tau,t)\mu(t)dt, \qquad (2.11)$$

$$(K\mu)(\tau) = \int_0^{2\pi} \frac{1}{2\pi}\cot\frac{\tau-t}{2}\mu(t)dt, \qquad (2.12)$$

The integrals in (2.11) and (2.12) are principal value integrals. The operator K is known as the conjugation operator. It is also known as the Hilbert transform (Henrici, 1986). The operators N, M, M_1 and K are bounded in H^{α}

and map H^{α} into H^{α} (Wegmann *et al.*, 2005). It follows from (2.7) that

$$M = M_1 - K. \tag{2.13}$$

In our study we consider only the generalized Neumann kernel with $A = \eta$.

Theorem 2.2 (Wegmann *et al.*, 2005)
Let N be the generalized Neumann kernel formed with A = η. Then $\lambda = 1$ is not an eigenvalue N. Let $w = R(z)$ maps Ω conformally onto $|w| < 1$ and $\Omega \cup \partial\Omega$ continuously onto $|w| \le 1$. Then, the Green function for Ω is given by (Henrici, 1986)

$$g(z, z_0) = -\frac{1}{2\pi}\log\left|\frac{R(z) - R(z_0)}{1 - R(z)R(z_0)}\right|. \tag{2.14}$$

In general the Green's function for Ω can be expressed by

$$g(z, z_0) = u(z) - \frac{1}{2\pi}\log|z - z_0|, z, z_0 \in \Omega, \tag{2.15}$$

where u is the unique solution of the interior Dirichlet problem

$$\begin{cases} \nabla^2 u(z) = 0, z \in \Omega \\ u(\eta(t)) = \log|\eta(t) - z_0|, \eta(t) \in \Gamma. \end{cases} \tag{2.16}$$

3. INTEGRAL EQUATION FOR THE INTERIOR DIRICHLET PROBLEM

Suppose that u is the unique solution of the interior Dirichlet problem (2.16). Since it is harmonic in Ω, u has a harmonic conjugate in Ω. We denote the boundary values of this harmonic conjugate by μ . Then $u + i\mu$ are boundary values of a function *f* analytic in Ω, i.e.,

$$f^{+}(\eta(t)) \rangle \; u(t) + i\mu(t), \eta(t) \in \Gamma, \tag{3.1}$$

where the " + " sign in the subscript denotes the boundary values from inside Ω. The function $f(z)$ is unique up to an additive imaginary constant which can be determined by assuming $f(0)$ is real. The following theorem gives an integral equation for μ .

Theorem 3.3 (Wegmann *et al.*, 2005)

Let μ be the unique solution of the integral equation

$$\mu - N\mu = -Mu, \tag{3.2}$$

where the kernels of the operators N and M are formed with $A = \eta$. Then the function $f^{+} = u + i\mu$ is a boundary value of an analytic function f in Ω with Im *f(0)= 0.*

By the Cauchy integral formula, the interior value of the function $f(z)$ stated above is given by

$$f(z) = \frac{1}{2\pi i}\int_{\Gamma}\frac{u + i\mu}{\eta - z}\,d\eta, \quad z \in \Omega. \tag{3.3}$$

By obtaining the unique analytic function f , the unique solution of the interior Dirichlet problem is given in $\Omega \cup \Gamma$ by

$$u(z) = \mathrm{Re} f(z). \tag{3.4}$$

4. NUMERICAL IMPLEMENTATION

The integral equation (3.28) in parameterized form is

$$\mu(s) - \int_0^{2\pi} N(s,t)\,\mu(t)\,dt =$$

$$-\int_0^{2\pi} M(s,t)\,\log|\eta(t) - z_0|dt \tag{4.1}$$

where the kernels N and M are formed with $A = \eta$. Let us denote the right-hand side of the equation (4.1) by $\psi(s)$. Then (4.1) becomes

$$\mu(s) - \int_0^{2\pi} N(s,t)\,\mu(t)\,dt = \psi(s) \tag{4.2}$$

where

$$N(s,t) = \begin{cases} \dfrac{1}{\pi}\mathrm{Im}\left[\dfrac{\eta(s)}{\eta(t)}\dfrac{\eta'(t)}{\eta(t) - \eta(s)}\right], & s \ne t, \\ \dfrac{1}{2\pi}\mathrm{Im}\left[\dfrac{\eta''(t)}{\eta'(t)}\right] - \dfrac{1}{\pi}\mathrm{Im}\left[\dfrac{\eta'(t)}{\eta(t)}\right], & s = t, \end{cases}$$

and

$$\psi(s) = -\int_0^{2\pi} M(s,t)\log|\eta(t)|\,dt$$

$$M(s,t) = -\frac{1}{2\pi}\cot\frac{s-t}{2} + M_1(s,t)$$

where

$$N(s,t) = \begin{cases} \dfrac{1}{\pi}\operatorname{Im}\left[\dfrac{\eta(s)}{\eta(t)}\dfrac{\eta'(t)}{\eta(t)-\eta(s)}\right], & s \neq t, \\[2ex] \dfrac{1}{2\pi}\operatorname{Im}\left[\dfrac{\eta''(t)}{\eta'(t)}\right] - \dfrac{1}{\pi}\operatorname{Im}\left[\dfrac{\eta'(t)}{\eta(t)}\right], & s = t, \end{cases}$$

Since the functions A and η are 2π - periodic, the integral operator N is discretized by the Nystrom method with trapezoidal rule (Atkinson, 1997).

Let n be a given integer and define the n equidistant collocation points t_j by

$$t_j = (j-1)\frac{2\pi}{n}, \quad j = 1,2,\ldots,n. \tag{4.3}$$

Then, using the Nyström method for (4.2) we obtain the linear system

$$\mu_n(t_i) - \frac{2\pi}{n}\sum_{j=1}^{n} N(t_i,t_j)\,\mu(t_j) = \psi(t_i) \tag{4.4}$$

where μ_n is an approximation to μ and

$$N(t_i,t_j) = \begin{cases} \dfrac{1}{\pi}\operatorname{Im}\left[\dfrac{\eta(t_i)}{\eta(t_j)}\dfrac{\eta'(t_j)}{\eta(t_j)-\eta(t_i)}\right], & t_i \neq t_j, \\[2ex] \dfrac{1}{2\pi}\operatorname{Im}\left[\dfrac{\eta''(t_j)}{\eta'(t_j)}\right] - \dfrac{1}{\pi}\operatorname{Im}\left[\dfrac{\eta'(t_j)}{\eta(t_j)}\right], & t_i = t_j, \end{cases}$$

and

$$\psi(t_i) = \frac{1}{2\pi}\int_0^{2\pi}\cot\frac{t_i - t}{2}\log|\eta(t)|\,dt - \frac{2\pi}{n}\sum M_1(t_i,t_j)\log|\eta(t_j)| \tag{4.5}$$

where

$$M_1(t_i,t_j) = \begin{cases} \dfrac{1}{\pi}\operatorname{Re}\left[\dfrac{\eta(t_i)}{\eta(t_j)}\dfrac{\eta'(t_j)}{\eta(t_j)-\eta(t_i)}\right] + \dfrac{1}{2\pi}\cot\dfrac{i-j}{2}, & t_i \neq t_j, \\[2ex] \dfrac{1}{\pi}\operatorname{Re}\left[\dfrac{1}{2}\dfrac{\eta''(t_j)}{\eta'(t_j)} - \dfrac{\eta'(t_j)}{\eta(t_j)}\right], & t_i = t_j. \end{cases}$$

The first term of the right-hand side of (4.4) can be calculated directly by using MATHEMATICA package, i.e., **Cauchy Principal Value**. Define the matrix $Q = [Q_{ij}]$ and vectors $\vec{x} = [x_i]$ and $\vec{y} = [y_i]$ by

$$Q_{ij} = \frac{2\pi}{n}N(t_i,t_j), x_i = \mu_n(t_i), y_i = \psi(t_i).$$

Next, the equation (4.2) can be written as an n by n system

$$(I - Q)\vec{x} = -\vec{y} \tag{4.6}$$

To solve the system (4.6) we will use the method of Gaussian elimination. Since (4.1) has a unique solution, then for a wide class of quadrature formula the system (4.6) also has a unique solution, as long as n is sufficiently large. After we get the unique solution $x_i = \mu_n(t_i)$, then we calculate $f_n(\eta(t_i))$ by using the following formula

$$f_n(\eta(t_i)) = \log|\eta(t_i) - z_0| + i\,\mu_n(t_i) \tag{4.7}$$

which represents the boundary values of $f(z)$ on the region Ω. By this result we can compute the interior values of $f(z)$ over the whole region Ω by using the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w-z}dw \tag{4.8}$$

But here in our work $w = \eta(t)$ and the integration runs from 0 to 2π, so (4.8) becomes

$$f_n(z) = \frac{1}{2\pi i}\int_0^{2\pi}\frac{f_n(\eta(t_j))}{\eta(t_j)-z}\eta'(t_j)\,dt \tag{4.9}$$

To increase the accuracy of $f_n(z)$ we shall use the following formula. Based on the fact that $\frac{1}{2\pi i}\int_\Gamma \frac{1}{\eta - z}d\eta = 1$, we can write $f_n(z)$ as

$$f_n(z)=\frac{\dfrac{1}{2\pi i}\displaystyle\int_0^{2\pi}\frac{f_n(\eta(t_j))}{\eta(t_j)-z}\eta'(t_j)\,dt}{\dfrac{1}{2\pi i}\displaystyle\int_0^{2\pi}\frac{\eta'(t_j)}{\eta(t_j)-z}dt} \qquad (4.10)$$

Then, using the Nyström method with the trapezoidal rule to discretize the integral in (4.10), we obtain the approximation

$$f_n(z)=\frac{\displaystyle\sum_{j=1}^{n}\frac{f_n(\eta(t_j))}{\eta(t_j)-z}\eta'(t_j)}{\displaystyle\sum_{j=1}^{n}\frac{\eta'(t_j)}{\eta(t_j)-z}} \qquad (4.11)$$

This has the advantage that the denominator in this formula compensates for the error in the numerator (Helsing and Ojala, 2008). Then, the real part of (4.11) $u_n(z)$ gives

$$u_n(z)=\operatorname{Re} f_n(z) \qquad (4.12)$$

Finally, by using $u_n(z)$ we can compute the Green's function $g_n(z,z_0)$ by the following formula

$$g_n(z,z_0)=u_n(z)-\log|z-z_0| \qquad (4.13)$$

5. NUMERICAL IMPLEMENTATION

In this example we consider oval of Cassini as simply connected region Ω as shown in Figure 4.1. The boundary of this region is parameterized by the function

$$\eta(t)=\left(\alpha^2\cos 2t+\sqrt{1-\alpha^4\sin^2 2t}\right)^{1/2}e^{it} \quad , \quad 0\le t\le 2\pi$$

$$|z-\alpha||z+\alpha|=1,\ 0<\alpha\le 1 \qquad ,z_0=0$$

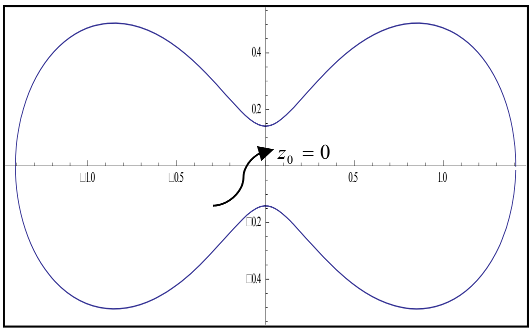

Fig. 4.1 The test region Ω with $\alpha = 0.99$.

The exact solution of this region Ω can be computed by using Riemann mapping function (Murid, 1997), i.e.

$$g(z,z_0)=-\frac{1}{2\pi}\log\left|\frac{R(z)-R(z_0)}{1-\overline{R(z_0)}R(z)}\right|$$

where

$$R(z)=\frac{1}{2\pi i}\int_0^{2\pi}\frac{e^{i\left(t-\frac{1}{2}\arg(w(t))\right)}}{\eta(t)-z}\eta'(t)\,dt,$$

$$\theta(t)=t-\frac{1}{2}\arg(w(t)),\ w(t)=\sqrt{1-\alpha^4\sin^2 2t}+i\,\alpha^2\sin 2t.$$

We describe the error by infinite-norm error $\|g(z,z_0)-g_n(z,z_0)\|_\infty$,where $g_n(z,z_0)$ is the numerical approximation of $g(z,z_0)$. We choose nine test points with $z_0=0$. The results are shown in Table 4.1.

Table 4.1 The error $\|g(z,z_0)-g_n(z,z_0)\|_\infty$.

n	32	64	128	256
0.5	0.001743	5.0439×10^{-5}	5.8000×10^{-8}	8.3324×10^{-11}
-0.5	0.001743	5.0439×10^{-5}	5.8002×10^{-8}	6.0659×10^{-11}
$1-0.2i$	0.000386	1.2565×10^{-5}	1.5242×10^{-8}	4.6438×10^{-11}
$1+0.3i$	0.000488	1.5927×10^{-5}	1.9143×10^{-8}	5.7295×10^{-11}
1	0.000291	9.6864×10^{-6}	1.1884×10^{-8}	3.6393×10^{-11}
$0.5-0.1i$	0.001626	4.7581×10^{-5}	5.4632×10^{-8}	8.9153×10^{-11}
-1	0.000291	9.6864×10^{-6}	1.1885×10^{-8}	3.5214×10^{-11}
$0.7-0.2i$	0.000411	9.6601×10^{-6}	9.9083×10^{-9}	1.7079×10^{-12}
$-1.2+0.1i$	0.000636	1.9045×10^{-6}	2.2356×10^{-8}	2.6771×10^{-11}

The levels curves of $g_n(z,z_0)$ are shown in Figure 4.2, while the 3D plot of the surfaces of $g_n(z,z_0)$ is shown in Figure 4.3.

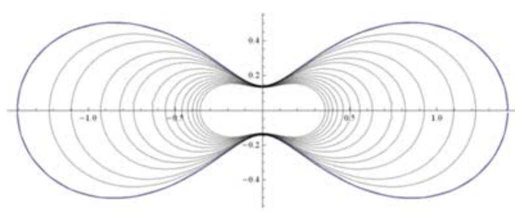

Fig. 4.2 Green's function for Ω in contours form.

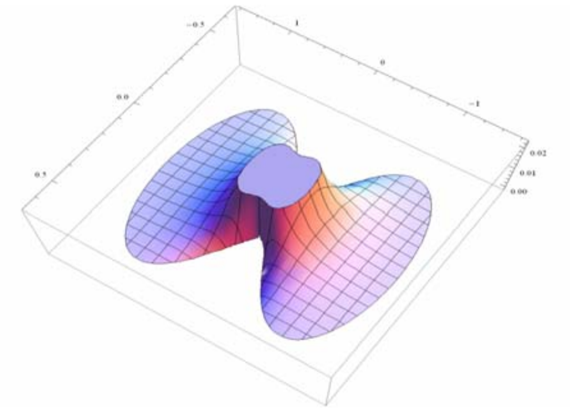

Fig. 4.3 Green's function for Ω in 3D form.

6. CONCLUSION

This study has presented a method for computing the Green's function on simply connected region by using a new approach based on boundary integral equation with generalized Neumann kernel. The idea for computing the Green's function on Ω is to solve the Dirichlet problem

$$\begin{cases} \Delta u(z) = 0, & z \in \Omega \\ u(\eta(t)) = \log\left|\eta(t) - z_0\right| \end{cases} \tag{5.1}$$

on that region by means of solving an integral equation numerically using Nyström method with the trapezoidal rule. Once we got the solution $u(z)$, the Green's function of Ω can be computed by using the formula

$$g(z, z_0) = u(z) - \log\left|z - z_0\right|. \tag{5.2}$$

The numerical example illustrates that the proposed method can be used to produce approximations of high accuracy.

For further research, we suggest the following:

- The domain of the problem in this paper is restricted to bounded simply connected region with smooth boundaries. We propose extending our approach to multiply connected regions and also to nonsmooth boundaries.
- In this paper, we have used MATHEMATICA package to calculate the first term of the right-hand side of the integral equation (4.6), which is a Cauchy principle value integral. Alternative approaches are the Fast Fourier Transform (FFT) or the Wittich's method.

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